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A Theoretical Exploration of the Flow about an Electric Arc Transverse to an Airstream using Potential Flow Methods

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A Theoretical Exploration of the Flow about an Electric Arc Transverse to an Airstream using Potential Flow Methods

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Summary.

The arc cross section is supposed to be a region in which heat is supplied to a continuous stream and in which a body force acts. Pressure variations are assumed small so that incompressible potential flow methods can be used to calculate streamlines and thence flow velocities of the compressible gas by a simple transformation. Example calculations are made in which a non-potential slit is used to obtain a positive drag coefficient, and flow diagrams are given. For future work it is suggested that some of the region constituting the arc should be separated from the external stream by an impenetrable boundary since this should enable closer agreement with experiment to be reached.

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1. *Introduction.*

It is sometimes necessary to supply heat to a high pressure gas stream, for subsequent expansion in a hypersonic wind tunnel for example, and electric arc heaters offer a promising way of doing this[1]. By its nature the heater must involve relative motion between the arc and the gas stream and one possible configuration is for the gas to flow in a direction transverse to the arc. One such arrangement[2] is for the arc to be struck across an annular gap between two concentric circular electrodes and made to rotate around the annulus by an imposed axial magnetic field. In addition, the gas flows axially through the annulus. In many such applications it is considered to be a reasonable approximation to regard most of the arc as a uniform column of conducting gas past each element of which the free stream flows in a two-dimensional manner[3]. In such a flow the arc loses heat by convection, which leads to the formation of a wake, and the viscous dissipation that accompanies this results in the arc having an aerodynamic drag[2], which in the steady state is opposed by electro-magnetic body forces.

The boundary conditions of the flow past the arc have been a matter of much conjecture[4]. Does the gas stream flow straight through the arc or does the arc behave something like a hot solid cylinder? If the latter, does the solid boundary coincide with the boundary of the conducting region, or is it quite different? The present Report attempts to throw some light on these questions by examining the effect of heat addition to the gas in a localised region. To do this the gas is assumed to be inviscid and non-conducting (both thermally and electrically) outside the region of heat supply. The addition of heat to a gas stream in circumstances of this sort has been considered before*[5], but it happens that in the case of a high pressure arc heater, the flow velocity is usually small, so that pressure variations are small compared with variations in specific volume. Thus throughout the flow region of interest

$$p\frac{D}{Dt}\left(\frac{1}{\rho}\right) \gg \frac{1}{\rho}\frac{Dp}{Dt} \tag{1}$$

and if the right hand side of (1) is neglected in the energy equation, a particularly simple form results, namely

$$\operatorname{div} \underline{v} = G(x,y)\,. \tag{2}$$

Here

p is the gas pressure,

t the time,

ρ the gas density,

and

$\underline{v}$ the flow velocity – a vector in the (x,y) plane normal to the arc axis, (*see* Fig. 1)

G is a function concerned with the supply of heat to the gas which is zero outside the cross section of the arc.

Neglecting the right hand side of equation (1) is equivalent to assuming that the speed of sound is infinite, often referred to as an assumption of incompressibility. In the present Report, however, the term incompressible is reserved for flows in which there is no density change, even though the sound speed is infinite; the assumption of infinite sound speed applies throughout the Report.

For practical arc heaters it may turn out that large velocities of the arc around the annulus are desirable, and Mach numbers approaching or even exceeding one may be used; in such circumstances the approximation (2) would need to be re-examined.

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*A recent survey of flows with heat addition is given in Ref. 6, but the flow past an electric arc as considered here represents a rather special case.

The form of equation (2) suggests that there is an analogy between the flow field through an arc and that of an incompressible fluid through a region of sources. By making use of this analogy, a solution to the compressible flow problem of a gas passing through an arc can be found in terms of the solution to the incompressible flow problem where the arc is replaced by the region of sources. In principle the method of solution is one of iteration, although this may not be a serious limitation for the purpose of obtaining general flow characteristics.

2. *Flow Equations for small Velocities.*

The steady state flow equations for a compressible inviscid gas subjected to body forces may be written,

continuity

$$\operatorname{div} \rho \underline{v} = 0 \tag{3}$$

momentum

$$\rho\,(\underline{v}\ \mathrm{grad})\,\underline{v} = -\mathrm{grad}\, p + \underline{f} \tag{4}$$

energy

$$\frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \frac{Q}{\rho} - \frac{1}{\rho}\operatorname{div} \underline{q} \tag{5}$$

where $\underline{f}$ is the body force per unit volume, Q is the heat addition per unit volume — *not* per unit mass as more commonly used

e is the internal energy of the gas per unit mass

$\underline{q}$ is the heat flow vector

Now for a perfect gas

$$\frac{De}{Dt} = \frac{1}{\gamma - 1}\left\{p\frac{D}{Dt}\left(\frac{1}{\rho}\right) + \frac{1}{\rho}\frac{Dp}{Dt}\right\} \tag{6}$$

where γ is the ratio of specific heats, and the second term on the right of equation (6) will be small compared with the first term for practical arcs with small flow velocity. The temperature ratio between the centre of the arc and the incident free stream, for example, may be as much as 100 with a comparable ratio in specific volume, whereas the pressure ratio will be everywhere near unity. Physically this means that the energy equation is almost entirely an equation of heat flow at constant pressure with negligible contribution from the kinetic energy of the flow. It also implies a low Mach number of the mainstream.

In the present Report, the last term in equation (6) is neglected and considerable simplification results. At the same time it is convenient to write

$$\underline{q} = -k\,\mathrm{grad}\, T \tag{7}$$

where k is the coefficient of thermal conductivity and T the absolute temperature. Since k may be assumed a function of T only, without serious error, equation (7) may be written

$$\underline{q} = -\,\mathrm{grad}\,\Phi \tag{8}$$

where Φ is the heat flux potential. Equation (5) now becomes, with $\frac{\partial}{\partial t} \rightarrow 0$,

$$\frac{\gamma}{\gamma-1} p(\underline{v} \text{ grad})\left(\frac{1}{\rho}\right) = \frac{Q}{\rho}+\frac{1}{\rho}\nabla^2 \Phi . \tag{9}$$

From the continuity equation (3), however, $(\underline{v} \text{ grad})\, \rho = -\rho \text{ div } \underline{v}$ so that (9) may be written

$$\text{div } \underline{v} = \frac{(\gamma-1)}{\gamma p}(Q+\nabla^2 \Phi) \equiv G(x,y) . \tag{10}$$

For the purpose of analysis we suppose that $G(x,y)$ is zero outside the arc, so that the arc consists of an inner electrically conducting core where Q and $\nabla^2 \Phi$ are both non-zero, surrounded by a thermal boundary layer (which is very thin in practice) where Q, but not $\nabla^2 \Phi$, is zero. The problem is, then, to find a flow field that satisfies equations (3), (4) and (10).

3. *Proposed method of solution.*

Consider the equations

$$\left.\begin{aligned} \rho_0\, (\underline{v}_i \text{ grad})\, \underline{v}_i &= -\text{grad } p_i + \underline{F}_i \\ \text{div } \underline{v}_i &= G_i(x,y) \end{aligned}\right\} \tag{11}$$

where ρ_0 is a constant and the suffix i is used to denote that the equations (11) relate to the flow problem of an incompressible fluid. In fact the problem is very like that of flow over a source distribution $G_i(x,y)$ in the presence of body forces $\underline{F}_i$ where $\underline{F}_i$ and G_i are both zero outside the arc, as is $\underline{f}$ in equation (4). There is, however, a difference in the momentum equation, to be discussed in detail later.

For the purpose of solving equations (11) in the examples used later the force distribution $\underline{F}_i$ will be assumed to be conservative such that curl $\underline{F}_i = 0$ and an irrotational solution can be found from a velocity potential. This puts a restriction on the nature of the body force distribution $\underline{f}$ for which a solution is obtained, but it is not essential to the method that $\underline{F}_i$ should be conservative provided that a solution to equations (11) can be found for $\underline{v}_i$ and p_i.

If equations (11) have been solved it remains necessary to adapt the solution to satisfy equations (3), (4) and (10). For the flow outside the arc a physical argument is helpful in suggesting a method of doing this. We suppose that $\underline{v}$ has everywhere the same direction as $\underline{v}_i$ so that the streamlines are unchanged in turning the incompressible solution into a compressible solution. Then since div $\underline{v} = 0$ in this region $\underline{v}$ grad $\rho = 0$, by (3) so that ρ is constant along the streamlines. This leads to the following picture:

(1) stream tubes that do not pass through the arc contain incompressible flow;

(2) stream tubes that pass through the arc are fattened in the process (by efflux from the sources in the case of $\underline{v}_i$) so that their continuation downstream of the arc contains fluid at a new constant density such that

$$\rho_1(uA)_1 = \rho_2(uA)_2 = \text{constant} \tag{12}$$

where A is the area of the stream tube, u here is the streamwise velocity $|\underline{v}|$, and the suffices 1 and 2 denote constant values upstream and downstream of the arc respectively.

It remains to satisfy the momentum equation (9). This can be done (since $\underline{f}$ and $\underline{F}_i$ are zero outside the arc) with

$$p = p_i \tag{13}$$

provided the momentum flow along a stream tube is the same for $\underline{v}$ as for $\underline{v}_i$, i.e.

$$\rho \underline{v}^2 A = \rho_0 \underline{v}_i^2 A \tag{14}$$

where ρ is the density at upstream infinity, and hence throughout the incompressible solution.
From (14)

$$\underline{v} = \underline{v}_i \sqrt{\frac{\rho_0}{\rho}}. \tag{15}$$

Mathematically the result (15) with $p = p_i$ follows at once from the momentum equations in streamwise co-ordinates (s,n), *viz*

$$\left.\begin{aligned} \rho u \frac{\partial u}{\partial s} &= -\frac{\partial p}{\partial s} \\ \frac{\rho u^2}{R} &= -\frac{\partial p}{\partial n} \end{aligned}\right\} \tag{16}$$

since $\frac{\partial p}{\partial s} = 0$.

Thus the solution given by (13) and (15) certainly satisfies the compressible flow equations (3), (4) and (10) outside the arc, and can clearly be made to satisfy the same equations inside the arc by suitable choice of $\underline{F}_i$ and G_i. Substitution of (13) and (15) together with (11) in equations (4) and (10) result in the following expressions for $\underline{F}_i$ and G_i after some simplification

$$\left.\begin{aligned} \underline{F}_i &= \underline{f} - \rho_0 \underline{v}_i \operatorname{div} \underline{v}_i = \underline{f} - \rho_0 \underline{v}_i G_i = \underline{f} - \underline{v}_i \frac{\sqrt{\rho\rho_0}}{2} G \\ G_i &= \frac{1}{2}\sqrt{\frac{\rho}{\rho_0}}\, G \end{aligned}\right\} \tag{17}$$

where G is given by (10).

To obtain a complete solution of the compressible flow equations for known $\underline{f}$ and G would thus require the following iteration procedure

(i) guess a distribution of $\underline{v}_i$ and ρ within the arc

(ii) determine $\underline{F}_i$ and G_i from (17)

(iii) solve equations (11) for $\underline{v}_i$ and p_i

(iv) determine ρ: equation (12) may now be generalised, since equation (15) applies both inside and outside the arc, to

$$\sqrt{\rho\rho_0}\, A\, u_i = \text{constant}, \tag{18}$$

where equation (18) applies along a stream tube and $u_i = |\underline{v}_i|$

(v) use the new values of ρ and $\underline{v}_i$ as a better approximation and repeat steps (i) to (iv) until a satisfactory agreement is obtained. Then derive $\underline{v}$ from equation (15).

In practice, however, the body force distribution $\underline{f}$ is not well known for the inside of an arc, and on the other hand a reasonable guess of the density distribution can probably be made, so a more useful process at the present stage seems to be to solve the equations for reasonable distributions of G_i and F_i and examine the resulting flow fields for the different values of the corresponding body force distribution.

4. *The Momentum Equation in the Source flow.*

It is of interest to derive the momentum equation from first principles for a flow in which a source distribution is present. Consider the x-momentum balance in the parallelepiped of Fig. 2 with sides parallel to the co-ordinate axes. There results

$$\left(\rho+\frac{\partial \rho}{\partial x}dx\right)\left(u+\frac{\partial u}{\partial x}dx\right)^2 dy\,dz-\rho u^2\,dy\,dz$$

$$+\left(\rho+\frac{\partial \rho}{\partial y}dy\right)\left(v+\frac{\partial v}{\partial y}dy\right)\left(u+\frac{\partial u}{\partial y}dy\right)dxdz-\rho uv\,dxdz+\frac{\partial p}{\partial x}dxdydz=f_x\,dxdydz \tag{19}$$

where f_x is the body force per unit volume in the x-direction. Equations similar to (19) follow for the y- and z-momentum, and the combined result may be written

$$\rho\,(\underline{v}\text{ grad})\,\underline{v}+\underline{v}\text{ div }\rho\underline{v}+\text{grad }p=\underline{f}. \tag{20}$$

In the absence of sources the term $\underline{v}$ div $\rho\underline{v}$ vanishes by continuity, but where mass sources are present div $\rho\underline{v}$ is finite.

It is not strictly necessary to cast equations (11) into the form appropriate to mass sources given by (20). The solution of equations (11) is only a step towards solution of (3), (4) and (10) and provided this can be successfully accomplished, there is no need to obtain a complete physical interpretation of the intermediate solution. On the other hand, a physical picture is often useful, and this can be retained for the intermediate solution by writing the first of equations (11) as

$$\left.\begin{aligned}\rho_0\,(\underline{v}_i\text{ grad})\,\underline{v}_i+\text{grad }p+\rho_0\,\underline{v}_i\text{ div }\underline{v}_i&=\underline{F}_i+\underline{F}_s\\ \text{where}\qquad \underline{F}_s&=\rho_0\,\underline{v}_i\text{ div }\underline{v}_i\end{aligned}\right\} \tag{21}$$

i.e. $\underline{F}_s$ represents the additional body force necessary to hold the sources in position as a result of the momentum $\rho_0\,\underline{v}_i$ div $\underline{v}_i$ that they give to the free stream. This force has a positive component in the x-direction indicating that the sources exhibit a negative drag.

It is interesting to note from equation (17) that

$$\underline{F}_i=\underline{f}-\underline{F}_s. \tag{22}$$

This implies that the body force $\underline{f}$ is the same as the total body force $\underline{F}_i+\underline{F}_s$ in the incompressible source flow, and in particular that the solution of the problem of heat addition without externally applied forces (for small pressure changes) can be derived from the solution of the problem of mass sources in an incompressible flow also without externally applied force, since the momentum equation is given by (21) with $\underline{F}_i+\underline{F}_s=0$. This last equation is not, however, satisfied in general by potential flow since vorticity can be introduced by the term $\rho_0\,\underline{v}_i$ div $\underline{v}_i$.

4.1. *Comparison with a Classical Result.*

It is well known (*see*, for example, Prandtl[7]) that a single point source of strength S (i.e. issuing S units of volume of fluid per unit time) experiences a thrust of magnitude $\rho_0\,S\,u_0$ if held in potential flow with a free stream velocity of u_0. The same result can be obtained from equation (20) by considering a uniform source distribution div $\underline{v}_s$ enclosed within a sphere. In potential flow, superposition of the velocity field ($\underline{v}_s$) due to the sources on that due to the free stream ($\underline{u}_0$) results in a flow field that satisfies

$$\rho\,(\underline{v}\text{ grad})\,\underline{v}+\text{grad }p=0\,. \tag{23}$$

Hence, by (20), the total body force $\underline{F}$ is given by

$$\underline{F} = \int_{\text{vol}} \rho_0 (\underline{u}_0 + \underline{v}_s) \operatorname{div} \underline{v}_s$$

$$= \rho_0 \operatorname{div} \underline{v}_s \int_{\text{vol}} (\underline{u}_0 + \underline{v}_s)$$

$$= \rho_0 \underline{u}_0 \operatorname{div} \underline{v}_s \,(\text{vol of sphere}) \tag{24}$$

since by symmetry $\int \underline{v}_s = 0.$

Hence

$$\underline{F} = (\rho_0 \, S \, u_0, 0, 0) \tag{25}$$

in agreement with the classical result.

5. *Example Solutions.*

Analytical solutions have inevitable limitations, and for detailed comparison with experiment it may be that numerical methods will need to be used. However, analytical solutions are often helpful in the early stages of an investigation and a few simple cases are discussed here.

The incompressible flow past the mass sources may be represented by means of a velocity potential ϕ defined by the relation,

$$u = -\frac{1}{a}\frac{\partial \phi}{\partial x}, \qquad v = -\frac{1}{a}\frac{\partial \phi}{\partial y}. \tag{26}$$

For simplicity in these examples, the arc is assumed to have a circular outer boundary of radius a; the origin is taken at the centre of the circle with x in the direction of the free stream, and x,y are made non-dimensional with respect to a. Non-dimensional polar co-ordinates (r, θ) are also used so that $r^2 = x^2 + y^2$ and the outer boundary of the arc is given by

$$r = 1\,. \tag{27}$$

It is also assumed that two forms of the velocity potential exist, ϕ_1 and ϕ_2 say, where ϕ_1 covers the region $r^2 > 1$, and ϕ_2 the inner region $r^2 < 1$. It is required that $\phi, \frac{\partial \phi}{\partial r}$ and $\frac{\partial \phi}{\partial \theta}$ are all continuous at $r = 1$, although in the more complicated example of Section 5.2. a discontinuity in velocity is allowed within the arc in order to represent a positive drag.

The total drag force is given by $-f_{xT}$ where

$$f_{xT} = a^2 \int_0^1 \mathrm{rdr} \int_0^{2\pi} f_x \, d\theta = a^2 \int_{\text{arc}} f_x \, dS \tag{28}$$

and where dS is a non-dimensional element of area. From (22)

$$f_x = F_{ix} + \rho_0 \, u_i \operatorname{div} v_i$$

$$= \frac{1}{a}\frac{\partial p}{\partial x} + \frac{\rho_0}{a^3}\left\{\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial \phi}{\partial x}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)\right\} \tag{29}$$

by (11). To simplify (28) and (29) we can make use of the continuity of pressure and obtain

$$f_{xT} = \frac{\rho_0}{a} \int\limits_{\text{arc}} \left\{ \frac{\partial \phi_2}{\partial x} \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial \phi_2}{\partial y} \frac{\partial^2 \phi_2}{\partial x \partial y} - \frac{\partial \phi_1}{\partial x} \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial \phi_1}{\partial y} \frac{\partial^2 \phi_1}{\partial x \partial y} + \frac{\partial \phi_2}{\partial x} \left(\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} \right) \right\} dS$$

$$= \frac{\rho_0}{2a} \int\limits_{\text{boundary}} \left\{ \left(\frac{\partial \phi_2}{\partial x} \right)^2 - \left(\frac{\partial \phi_1}{\partial x} \right)^2 + \left(\frac{\partial \phi_2}{\partial y} \right)^2 - \left(\frac{\partial \phi_1}{\partial y} \right)^2 \right\} dy + \frac{\rho_0}{a} \int\limits_{\text{arc}} \frac{\partial \phi_2}{\partial x} \left(\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} \right) dS. \tag{30}$$

It follows that if there are no discontinuities in velocity within the arc, the first term in (30) vanishes by virtue of the boundary condition at $r = 1$. The integrand in the second term in (30) is the product u div v and it will be convenient to take examples in which only one term in the velocity potential, say ϕ_{2h}, makes a net contribution to div v, i.e. to the heat supplied to the airstream. The second integral in (30) may be integrated to give

$$\frac{\rho_0}{2a} \int\limits_{\text{boundary}} \left\{ \left(\frac{\partial \phi_2}{\partial x} \right)^2 - \left(\frac{\partial \phi_2}{\partial y} \right)^2 \right\} dy + \frac{\rho_0}{a} \int\limits_{\text{boundary}} \frac{\partial \phi_2}{\partial y} \frac{\partial \phi_2}{\partial x} dx$$

so that there is no contribution to f_{xT} from terms for which the velocity vanishes at $r = 1$. In addition, doublet-like terms produce no contribution by reasons of symmetry so that f_{xT} depends only on the product of the heat supply term and the uniform velocity term in ϕ_2. The net result is the same as that given in Section 4.1., that $f_{xT} = \rho_0 u_0 S$ where S is the total increase in volume of the fluid in unit time per unit length of arc. The transformation to compressible flow does not affect this result because the local momentum and pressure gradients are unaffected by the transformation so that the total body force as deduced from the external flow field must be unaltered.

The preceding argument shows that the drag of the arc will always be negative as long as the flow field is deduced wholly by potential methods. In reality, however, the wake originates in viscous flow that induces rotation and leads to a positive drag, so that a region must be introduced where the flow is not of potential type. For mathematical convenience the region is condensed into an infinitesimal slit along the y-axis across which the velocity is discontinuous and f_{xT} is now given by integration over the two semi-circles, $x > 0$ and $x < 0$. This is equivalent to locating a line of sources along $x = 0$ but discarding the f_x that would accompany them in potential flow; then if the line source is made stronger than the net source for the whole arc it follows that the potential part of ϕ_2 contributes a net sink and a positive drag*. This method is applied in the example of Section 5.2. where the line source can be regarded as being included with a ϕ_2 term that extends over the semicircle for $x > 0$ and for which $\int \nabla^2 \phi_2$ including the line source is zero; thus the net heat addition is still given by the term ϕ_{2h}. Physically the significance of this is that all the viscous effects that bring about a positive drag are supposed to take place in the slit at $x = 0$ across which there is a discontinuous pressure drop and increase in velocity. This method of representing the effects of viscous drag is illustrated by Küchemann and Weber[8]; an alternative would be to introduce rotation by a distribution of vortices in the wake, or as a limiting case, by two line vortices along the wake boundary. In addition, however, the application in 5.2. contains a certain arbitrariness owing to the interchangeability between pressure gradient and body force within the arc;

*Actually a much weaker line source is sufficient, as in Section 5.2., if the velocity at $x = 0$ is high.

this interchangeability has no net effect as long as the flow is wholly potential, but when a non-potential slit is used the drag and overall pressure drop are affected as discussed in 5.2. below. First, however, we consider simple examples with no slit.

5.1. *Solutions with Symmetric Velocity Potential.*

Apart from the uniform velocity term, the velocity potentials considered here are symmetric in x and y so that the corresponding velocities are anti-symmetric. For the outer flow ($r \geqslant 1$)

$$\phi_1 = -u_0\, ar\cos\theta - u_h\, a\log r\,, \tag{31}$$

and for the inner flow ($1 \geqslant r \geqslant 0$)

$$\left.\begin{aligned} \phi_{2a} &= -u_0\, ar\cos\theta + \tfrac{1}{2}u_h\, a(1-r^2) - u_1\, ar^2(1-r^2)^2\cos^2\theta \\ \phi_{2b} &= -u_0\, ar\cos\theta + \tfrac{1}{4}u_h\, a(1-r^2)(3-r^2) - u_1\, ar^2(1-r^2)^3\cos^2\theta. \end{aligned}\right\} \tag{32}$$

The first term in ϕ_1 represents the free stream with velocity $(u,v) = (u_0,0)$ and the second term represents the flow away from a symmetric source distribution within the circle $r = 1$. ϕ_{2a} and ϕ_{2b} are possible matching solutions where the third term and its derivatives vanish at $r = 1$. The corresponding source distributions are given by

$$G_i = \operatorname{div} \underline{v}_i = -\frac{1}{a^2}\nabla^2\phi = -\frac{1}{a^2}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) \tag{33}$$

where ∇^2 is non-dimensional. From (32)

$$\left.\begin{aligned} -\nabla^2\phi_{2a} &= 2a\,u_h + 2a\,u_1\{(1-r^2)^2 - 4r^2(3-4r^2)\cos^2\theta\} \\ -\nabla^2\phi_{2b} &= 4a\,u_h(1-r^2) + 2a\,u_1(1-r^2)\{(1-r^2)^2 - 6r^2(3-5r^2)\cos^2\theta\} \end{aligned}\right\} \tag{34}$$

The basic difference between ϕ_{2a} and ϕ_{2b} is seen to be that in ϕ_{2b} the source strength vanishes at $r = 1$ as it should in practice, since it is supposed to include the effects of thermal conduction. The total mass outflow upon which depends the total heat supply to the airstream by (17) and (10), is given by the integral of G_i over the region $r < 1$ and equals $2\pi a\, u_h$ in each case.

The term in u_1 is introduced to allow a variation of the distribution of body force within the arc. This is possible because $-\nabla p$ and f_i are interchangeable in the momentum equation, and because the term vanishes at $r = 1$. Two extreme assumptions are possible: that $f_i = 0$ in which case $f_x = u \operatorname{div} \underline{v}$ everywhere, or that $\nabla p = 0$, in both cases for the u_i term only. By either assumption the pressure is the same at $r = 1$, since the flow is wholly potential, and for the same reason the total drag is the same, although neither of these results is true in the next example. The drag may be calculated entirely from the external stream, since it is given by the sum of the pressure drop in the x direction together with the fall in x-momentum, both integrated around $r = 1$. By Fig. 1

$$\text{drag} = a\int_{-1}^{1} dy\,(p_l - p_r) + a\rho_0\int_{-1}^{1} dy\,(u_l^2 - u_r^2) + a\rho_0\int_{-1}^{1} dx\,(u_b v_b - u_t v_t)$$

$$= \tfrac{1}{2}a\rho_0\int_{-1}^{1} dy\,(u_r^2 + v_r^2 - u_l^2 - v_l^2) + a\rho_0\int_{-1}^{1} dy\,(u_l^2 - u_r^2) + a\rho_0\int_{-1}^{1} dx\,(u_b v_b - u_t v_t)$$

$$= \tfrac{1}{2}a\rho_0\int_{-1}^{1} dy\,(u_l^2 - u_r^2 + v_r^2 - v_l^2) + a\rho_0\int_{-1}^{1} dx\,(u_b v_b - u_t v_t)\,. \tag{35}$$

From (32)

$$\left.\begin{aligned} u_a &= u_0+u_h\,x+2\,u_1x(1-r^2)(1-r^2-2x^2) \\ u_b &= u_0+u_h\,x(2-r^2)+2\,u_1x(1-r^2)^2(1-r^2-3\,x^2) \\ v_a &= u_h\,y-4\,u_1\,x^2y(1-r^2) \\ v_b &= u_h\,y(2-r^2)-6\,u_1\,x^2\,y(1-r^2)^2 \end{aligned}\right\} \quad 0 \leqslant r \leqslant 1 \tag{36}$$

and from (31)

$$\left.\begin{aligned} u &= u_0+u_h\,x/r \\ v &= u_h\,y/r\,. \end{aligned}\right\} \quad 1 \leqslant r \tag{37}$$

Substitution in (35) gives, as expected

$$\text{drag} = -2\pi\,\rho_0\,u_0\,u_h\,. \tag{38}$$

It may be verified that $\int \operatorname{div} \underline{v} = 0$ for the term in u_1, a result which follows also from the fact that the flow due to u_1 across $r = 1$ vanishes everywhere.

The two examples considered here are therefore unrealistic in that they relate to arcs with negative drag and a further lack of realism is indicated by the pattern of stream lines illustrated in Figs. 3 and 4. The stream lines are the same for the compressible flow as for the incompressible flow, and the lack of realism referred to is concerned with the way in which the flow leaves the arc on the downstream side. Equations (34) show that the heat supply to the airstream approaches zero at $r = 1$ from an essentially positive value so that the temperature of the gas is still increasing as it leaves the arc, whereas in reality it must fall in this region as the gas becomes less electrically conducting. These two shortcomings are to some extent removed in the next example.

It can be seen from Fig. 5 that if the u_h term is made large enough a stagnation point develops in the flow and the external flow may not reach the arc boundary at all. When this happens there exists, in the incompressible flow, a wake behind the stagnation streamline that never closes and is such that the streamlines in this region all originate from a single point as in the case of a point source but with the difference that the flow velocity there is zero rather than infinite. Again flow diagrams of this type are unrealistic because they require the creation of fluid behind the stagnation streamline (it is clearly no longer possible for this creation to be replaced by a density change in transforming to the compressible flow because the external flow does not penetrate to this region at all) but it turns out that they are never in fact needed to represent strong heat supply in this type of example. The velocity u has the value u_0-u_h at the point $x = -1$, so that if the net source strength is increased from zero a stagnation point will be produced at $x = -1$ for $u_h = u_0$. Before this happens, however, the stream tubes will expand in area as they pass through the arc to an ever increasing extent so that there is no limit to the drop in density that can be obtained from equation (18). There is therefore no limit to the heat source strength G, given by equation (17) with G_t fixed, that can be obtained without introducing a stagnation point solution. Fig. 3c is an illustration of flow that has no stagnation point although it is very near to forming one, and shows a large expansion: in Fig. 4c a stagnation point has just formed. The flow solutions with stagnation points are not wholly absurd because the creation of fluid could represent approximately the condition of small axial flow along the arc with a steady feed from this flow into the wake. The equations in this case are discussed in the Appendix.

5.2. *Solution with a Non-potential Velocity Jump.*

As explained earlier a positive drag may be obtained by introducing a slit in the potential flow at $x = 0$ across which there is a velocity increase. This is done by introducing a u_1 term with the condition that $u_1 = 0$ for $x < 0$; the term chosen is different from those used in the last example since it is necessary that the coefficient of u_1 should not vanish at $x = 0$. In addition a heat sink effect (caused, for example, by radiation loss) is introduced on the downstream side of the arc so that the streamlines are closing in this region. This calls for a matching term of doublet type in the external flow, and the resulting velocity potential is given by,

$$\phi_1 = -u_0\, ar\cos\theta - u_h\, a\log r + (u_2\, a\cos\theta)/r \qquad r \geqslant 1 \tag{39}$$

$$\phi_2 = -u_0\, ar\cos\theta + \tfrac{1}{4}u_h a(1-r^2)(3-r^2) + u_1\, ar\cos\theta(1-r^2)^3 + u_2\, ar\cos\theta(3-3r^2+r^4), \qquad 0 \leqslant r \leqslant 1 \tag{40}$$

where $u_1 = 0$ for $x < 0$.

In practice, u_0 and u_h are both positive, u_1 and u_2 both negative. The corresponding velocities are

$$\left.\begin{aligned} u &= u_0 + u_h\, x/r^2 + u_2(x^2-y^2)/r^4 \\ v &= u_h\, y/r^2 + 2\, u_2\, xy/r^4 \end{aligned}\right\} \quad r \geqslant 1 \tag{41}$$

$$\left.\begin{aligned} u &= u_0 + u_h\, x(2-r^2) - u_1\, s^2(s-6x^2) - u_2\{3s^2+3\,sr^2+r^4-x^2(6s+2r^2)\} \\ v &= u_h\, y(2-r^2) + 6\, u_1\, s^2\, xy + u_2\, xy(6s+2r^2) \end{aligned}\right\} \quad 0 \leqslant r \leqslant 1 \tag{42}$$

where $s = 1-r^2$. Finally, the source strength G_i is given by

$$a^2\, G_i = -\nabla^2\phi = 4au_h\, s - 24\, u_1\, axs(r^2-s) + 24\, u_2\, axs. \tag{43}$$

If u_2 is negative, equation (43) shows that this term provides a heat sink in $x > 0$ and a heat source in $x < 0$ with no net source strength. The term in u_1 integrated over the region $x > 0$, $r \leqslant 1$ gives a contribution to $\int G_i$ of

$$-24\, u_1\, a \int_0^1 r\, dr \int_{-\pi/2}^{\pi/2} rs(r^2-s)\cos\theta\, d\theta = \frac{32}{35}u_1\, a \tag{44}$$

and with u_1 negative this represents a net sink. Since u_1 does not appear in the external flow, it follows that the discontinuity at $x = 0$ is equivalent to a net source of $-\frac{32}{35}\, u_1 a$ distributed along the line $x = 0$, as may be verified directly from the velocity jump. Locally, however, near $r = 1$ on the downstream side of the arc the contribution of a negative u_1 is that of a heat source and in order that this shall not override the effect of u_2 in this region it is desirable that $(4\, u_h + 24|u_1|x - 24|u_2|x)_{r=1}$ should be negative for most of the boundary in $x > 0$.

To estimate the drag we may again start with the potential flow drag given by (35), when substitution of (41) once more gives the expected result (38). However, as explained earlier, it is intended that any forces acting in the slit shall be excluded, since we do not want to represent potential flow in this region. This can be allowed for by using (35) across the slit, i.e.

$$\text{slit drag} \doteq a \int_{-1}^{1} dy(p_l - p_r) + a\rho_0 \int_{-1}^{1} dy(u_l^2 - u_r^2) \tag{45}$$

where the suffixes l and r refer to the values at $x = 0-$ and $x = 0+$ respectively. Only terms in u_1 contribute to (45), and the magnitude of their contribution depends on the assumption made about the pressure distribution associated with this term. If the two extreme assumptions referred to in 5.1. are denoted by

$$\left.\begin{array}{lll} \mathrm{I}, & \underline{f}_i = 0, & \nabla p \neq 0 \\ \mathrm{II}, & \nabla p = 0, & \underline{f}_i \neq 0 \end{array}\right\} \text{ for the } u_1 \text{ term}$$

then in I Bernoulli's equation holds and since $v = 0$ at $x = 0$ the first term in (45) becomes equal to half the second term in magnitude and opposite in sign. In II the first term in (45) vanishes, so we may write

$$\text{slit drag} = \tfrac{1}{2} k a\rho_0 \int_{-1}^{1} dy(u_l^2 - u_r^2) \tag{46}$$

where k has the value 1 in I and 2 in II.

So far (38) and (46) assume wholly potential flow but we now destroy the potential nature of the flow across the slit by discarding the slit drag. The net drag is given by subtracting (46) from (38), and after substitution of (42) in (46) this gives

$$\frac{\text{drag}}{a\rho_0} = -2\pi\, u_0\, u_h + k(-0{\cdot}914\, u_0\, u_1 + 0{\cdot}341\, u_1^2 + 2{\cdot}464\, u_1\, u_2)\,. \tag{47}$$

Or, if a drag coefficient is defined by

$$\text{drag} = \rho_0\, a\, u_0^2\, C_D \tag{48}$$

then

$$C_D = -2\pi\, \tilde{u}_h + k(-0{\cdot}914\, \tilde{u}_1 + 0{\cdot}341\, \tilde{u}_1^2 + 2{\cdot}464\, \tilde{u}_1\, \tilde{u}_2) \tag{49}$$

where $\tilde{u}_h$, $\tilde{u}_1$, and $\tilde{u}_2$ are non-dimensional values of u_h, u_1 and u_2. The same result may be obtained from f_{xT} where the integration is carried out over the two regions $x > 0$ and $x < 0$ but excluding the localised forces in the slit at $x = 0$.

Three cases are illustrated in Figs. 6, 7 and 8 for values of $\tilde{u}_h$, $\tilde{u}_1$ and $\tilde{u}_2$ given in Table 1 below.

TABLE 1

$\tilde{u}_h$	$\tilde{u}_1$	$\tilde{u}_2$	$C_D, k = 1$	$C_D, k = 2$
0·2	−0·4	−0·6	−0·245	0·767
0·2	−0·4	−1·0	0·150	1·56
0·2	−0·5	−0·68	0·124	1·50

The large variation in drag coefficient between the values for $k = 1$ and $k = 2$ makes it possible to choose any value in the practical range (say $C_D < 1$) by suitable choice of k. The effect of a positive drag coefficient appears in the external flow as a drop in pressure in the wake, where some further readjustment of the flow must take place. Figs. 6, 7 and 8 illustrate three different flow patterns. In Fig. 6 the flow passes directly through the arc and the streamlines show first a large expansion followed by a smaller contraction; numerical values for the centreline stream tube are shown on the diagram. These values are greatly exaggerated by comparison with any possible practical flow and correspond to enormous temperatures near the centre of the arc as shown by Table 2.

TABLE 2

x	y	$\tilde{u}_i$	$\tilde{v}_i$	ρ/ρ_0	T/T_0	$\tilde{u}$
$-2{\cdot}90$	$0{\cdot}101 \times 10^{-2}$	0·860	$7{\cdot}36 \times 10^{-5}$	1·0	1·0	0·860
$-0{\cdot}910$	$0{\cdot}990 \times 10^{-2}$	0·172	$1{\cdot}68 \times 10^{-2}$	0·258	3·88	0·339
$-0{\cdot}512$	$5{\cdot}04 \times 10^{-2}$	1·41	$9{\cdot}41 \times 10^{-2}$	$1{\cdot}49 \times 10^{-4}$	$6{\cdot}71 \times 10^{3}$	115
$-0{\cdot}213$	$6{\cdot}44 \times 10^{-2}$	2·47	$7{\cdot}28 \times 10^{-2}$	$2{\cdot}97 \times 10^{-5}$	$3{\cdot}37 \times 10^{4}$	454
$-0{\cdot}013$	$6{\cdot}84 \times 10^{-2}$	2·79	$3{\cdot}04 \times 10^{-2}$	$2{\cdot}01 \times 10^{-5}$	$4{\cdot}98 \times 10^{4}$	622
0·087	$6{\cdot}88 \times 10^{-2}$	3·15	$-8{\cdot}14 \times 10^{-3}$	$1{\cdot}60 \times 10^{-5}$	$6{\cdot}25 \times 10^{4}$	788
0·187	$6{\cdot}79 \times 10^{-2}$	2·96	$-4{\cdot}61 \times 10^{-2}$	$1{\cdot}87 \times 10^{-5}$	$5{\cdot}35 \times 10^{4}$	684
0·587	$5{\cdot}00 \times 10^{-2}$	1·25	$-9{\cdot}48 \times 10^{-2}$	$1{\cdot}93 \times 10^{-4}$	$5{\cdot}18 \times 10^{3}$	89·9
1·01	$1{\cdot}97 \times 10^{-2}$	0·606	$-1{\cdot}93 \times 10^{-2}$	$5{\cdot}15 \times 10^{-3}$	$1{\cdot}94 \times 10^{2}$	8·44
3·01	$1{\cdot}19 \times 10^{-2}$	1·00	$-2{\cdot}63 \times 10^{-4}$	$5{\cdot}29 \times 10^{-3}$	$1{\cdot}89 \times 10^{2}$	13·8

Although the absolute magnitude of the temperatures in Table 2 is excessive, the distribution is qualitatively correct, and on Fig. 6 the mean value of T/T_0 between the two outer streamlines shown rises to a peak of about 60. It is also possible to compare the overall heat supply with a practical value. Specimen laboratory measurements for an arc being driven between two parallel electrodes are, in m.k.s.a. units:

current = 1200 amp

electric field = 1700 volts/meter

lateral velocity = 100 m/sec

core radius = 0·0045 m

from which the heat supply, neglecting radiation, is 1200×1700 or $2{\cdot}04 \times 10^6$ joules/meter, sec.

From (10) and (17)

$$Q = \frac{\gamma p}{\gamma - 1} G = \frac{2\gamma p}{\gamma - 1} G_i \sqrt{\frac{\rho_0}{\rho}}. \tag{50}$$

For an order of magnitude comparison we may take a mean value of $\sqrt{\rho_0/\rho}$ based on the downstream and upstream values of uA taking the outermost streamline of Fig. 6. This gives a mean value for $\sqrt{\rho_0/\rho}$ of about 7, and then integrating over the circle for the u_h term of $-\nabla^2\phi$ gives

$$Q = \frac{2\gamma p}{\gamma - 1} \times 7 \times 2\pi\, a u_h = \frac{5{\cdot}6\gamma\pi}{\gamma - 1} pa\, u_0 \tag{51}$$

since $u_h = 0{\cdot}2\ u_0$. Substitution of the measured values given above together with $\gamma = 1{\cdot}4$ and $p = 10^5$ newtons/m^2 (atmospheric pressure) yields $Q = 2{\cdot}8 \times 10^6$ joules/meter, sec or something like twice what it should be allowing for radiation.

Fig. 7 is analogous to Fig. 5 where the external flow does not penetrate the arc at all and where an open wake persists downstream. Analysis of this case for a slow axial flow would be possible on the lines indicated in the Appendix. Fig. 8 is not different in principle from Fig. 6, but illustrates a flow pattern in which a stagnation point has very nearly been created but not quite. In the incompressible solution the swollen central region (shaded in Fig. 8) consists largely of fluid being created and then destroyed again a little way downstream and the analysis based on through flow would result in even more extreme temperatures than are given by Table 2. Fig. 8, however, does give a strong hint that more practical solutions could be found by introducing a region within $r = 1$ to be defined by an inner impenetrable boundary which in turn acts as a line heat source for the external convection. The reason for saying this is that although the shaded region of Fig. 8 together with its mirror image includes about 45 per cent of the arc cross-section a numerical integration shows that it includes only about $7\frac{1}{2}$ per cent of the net G_i. The net heat supplied to the compressible gas in the same region can be estimated directly from the stream lines of Fig. 8. The heat content in the wake is $\rho u\, C_p\, T$ per unit area, and to a first approximation (ρT), u and C_p are all constant, so that the heat content is directly proportional to the stream tube size. If we neglect the heat content of the flow upstream of the arc, this means that the shaded area of Fig. 8 receives rather more than 10 per cent of the total output from the arc. The net heat supply however, can be expressed as $\int G_i \sqrt{\dfrac{\rho_0}{\rho}}$ and in view of the very large value of $\sqrt{\rho_0/\rho}$ in the shaded area of Fig. 8 this integral amounts to the difference of two nearly equal quantities: thus only a very small change in the streamline pattern would be needed to enclose a large area with no net heat addition, and which could be represented by an impenetrable boundary enclosing recirculating flow. Once this had been done a more reasonable temperature distribution could be represented.

The present examples have not been pursued to the extent of evaluating current distributions since this would be premature, but in order to do so the method would be to calculate temperature (T) and density (ρ) distribution on the lines of Table 2 throughout the arc cross section. The distribution of G_i would be given by (43) and of G by (17). $\nabla^2\Phi$ follows from the temperature distribution, and Q from equation (10). Then $Q = EJ - Q_R$, where Q_R is the radiation loss calculable from ρ and T, and since the electric field E is constant over the (x,y) plane this would give the distribution of current density, J.

6. *Conclusions and Further Developments.*

The conclusions may be summarised.

(1) Solutions to the problem of flow of a gas through a region of heat supply at nearly constant pressure may be obtained from incompressible potential flow through a region of sources.

(2) The velocities downstream of the heat supply are related to those upstream by equation (18), *viz*

$$\sqrt{\rho\rho_0}\, A\, u_i = \text{constant}\,.$$

(3) If the region of heat supply arises from an electric arc, then the simplest potential flow solutions relate to arcs with a negative drag, and with maximum temperatures on the downstream periphery of the arc.

(4) By introducing doublet-like terms a more reasonable temperature distribution can be obtained, and by introducing a line singularity, arcs with positive drag can be represented.

(5) The solutions are extremely sensitive in the central regions and seem likely to imply impractically high temperatures for practical amounts of net heat supply.

Following conclusions (4) and (5) the most hopeful line of development seems to be to prescribe an inner impermeable boundary; otherwise the treatment would be similar to that in Section 5.2. The shape of the inner and outer boundaries might well come from spectroscopic measurements of temperature in the laboratory. Since it is unlikely that the measured boundaries would turn out to be circles the corresponding solutions would probably involve numerical rather than analytical methods.

A well known practical difficulty with high pressure arc discharges is that they are often unstable. The calculations of Section 5 leave an impression that this is not very surprising on the grounds of heat addition alone. A sudden local current excess might, for example, cause a temporary stagnation point to form and this could have a disproportionate effect on the flow. Another argument is that if the arc does contain a stagnation region (the inner boundary of Conclusion 5) then the dense external flow is diverted around the rarefied column of recirculating flow and in the neighbourhood of the stagnation point one might expect instabilities of the Rayleigh-Taylor type. These aspects deserve further study, but it is worth noting that if they do contain the root of the instabilities of an arc across an airstream, one might expect that a superimposed axial flow of the type discussed in the Appendix could be used to improve the stability. Practical arrangements for such a flow would, however, be difficult.

LIST OF SYMBOLS

A		Stream tube area
C_D		Drag coefficient
$\underline{E}$		Electric field
$\underline{F}$		Body force (equation (24))
$\underline{F}_i, \underline{F}_s$,		Body force (equation (22) also (17))
G, G_i		Heat supply and source strength (equation (10) and equation (17))
H	=	$h+\frac{1}{2}u^2$
$\underline{J}$		Electric current density
Q		Heat liberated/unit vol
$\bar{Q}$	=	$Q+\nabla^2\Phi$
S		Source strength (Section 4.1.)
T		Absolute temperature
a		Arc radius
e		Internal energy/unit mass
$\underline{f}$		Body force in compressible flow
h	=	$e+p/\rho$, enthalpy/unit mass
k	=	Thermal conductivity (equation (7))
m_c	=	div $\rho\underline{v}$
p		Pressure
$\underline{q}$		Heat flux vector
r		Co-ordinate (made non-dimensional by a in Section 5)
s		Distance along streamline
t		Time
u		Sometimes used for the streamwise component of $\underline{v}$, otherwise the x component
u_0, u_1, u_2, u_h		Variables in the velocity potential
$\underline{v}$		Velocity vector (u,v,w)
(x,y,z)		Rectangular co-ordinates, non dimensional by a in Section 4
Φ		Heat flux potential (equation (8))
θ		Angular co-ordinate
ρ		Density
ϕ		Velocity potential (ϕ_1 outer, ϕ_2 inner)

Non dimensional velocities are denoted by $(\tilde{u}, \tilde{v}, \tilde{w})$ if required

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APPENDIX

Two Dimensional Arcs with a Net Output of Fluid.

A.1. *Momentum and energy equations.*

Suppose that fluid is created within the arc column by mass sources so that div $\rho v \neq 0$. The momentum equation was derived for this case in the main text (equation (20)) and may be written in tensor notation

$$\rho u_i \frac{\partial u_j}{\partial x_i} + u_j \frac{\partial \rho u_i}{\partial x_i} + \frac{\partial p}{\partial x_j} = f_j . \tag{A1}$$

The energy equation follows in the same way, where we suppose that the created fluid particles possess the ambient enthalpy but no kinetic energy:

$$\frac{\partial}{\partial x_i}(\rho u_i H) - h \frac{\partial}{\partial x_i}(\rho u_i) = \bar{Q} + f_i u_i \tag{A2}$$

where $h = e + p/\rho$

$$H = h + \tfrac{1}{2} u^2, (u^2 = u_1^2 + u_2^2)$$

$$\bar{Q} = Q + \nabla^2 \Phi$$

Rewrite (A.2) as

$$\rho u_i \frac{\partial h}{\partial x_i} + \tfrac{1}{2} u^2 \frac{\partial \rho u_i}{\partial x_i} + \rho u_i u_j \frac{\partial u_j}{\partial x_i} = \bar{Q} + f_i u_i \tag{A3}$$

multiply (A1) by u_j and subtract from (A3) to give

$$\rho u_i \frac{\partial h}{\partial x_i} - \tfrac{1}{2} u^2 \frac{\partial \rho u_i}{\partial x_i} - u_i \frac{\partial p}{\partial x_i} = \bar{Q}$$

or

$$\rho u_i \frac{\partial e}{\partial x_i} + \rho u_i p \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\right) - \tfrac{1}{2} u^2 \frac{\partial \rho u_i}{\partial x_i} = \bar{Q} . \tag{A4}$$

But the assumption of small pressure variation implies that

$$\frac{\partial e}{\partial x_i} \approx \frac{p}{\gamma - 1} \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\right)$$

and hence (A4) becomes

$$\frac{\gamma}{\gamma - 1} \rho u_i p \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\right) - \tfrac{1}{2} u^2 \frac{\partial \rho u_i}{\partial x_i} = \bar{Q}$$

or, since

$$u_i \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{\partial \rho\, u_i}{\partial x_i}+\frac{1}{\rho}\frac{\partial u_i}{\partial x_i},$$

$$\frac{\gamma}{\gamma-1}p\frac{\partial u_i}{\partial x_i} = \bar{Q}+H\frac{\partial \rho\, u_i}{\partial x_i}. \tag{A5}$$

Alternatively if the created fluid is in fact produced by a slow drift in the z direction it is more realistic to return to the original equations of the text with div $\rho \underset{\sim}{v} = 0$ (these equations are valid in 3 dimensions) but to consider the flow within a thin slice ($z_1 < z < z_2$, say) within which the z component of velocity is negligibly small and all the other dependent variables are independent of z to sufficient accuracy. Then if w is the component of velocity in the z direction we have

$$w \ll u, v \tag{A6}$$

$$\frac{\partial \rho\, w}{\partial z}+\mathrm{div}_2\, \rho \underset{\sim}{v} = 0$$

or approximately

$$\frac{\partial w}{\partial z} = -\frac{1}{\rho}\mathrm{div}_2\, \rho \underset{\sim}{v} \tag{A7}$$

where div_2 stands for the two-dimensional divergence. The momentum equation in the two-dimensional (x,y) flow now returns to the form (4) in the main text, i.e.

$$\rho\,(\underset{\sim}{v}\,\mathrm{grad})\,\underset{\sim}{v} = -\mathrm{grad}\, p+\underset{\sim}{f} \tag{A8}$$

since the extra term $\rho w \dfrac{\partial \underset{\sim}{v}}{\partial z}$ is small. The energy equation is derived from (10) of the main text, i.e.

$$h\,\rho\, \mathrm{div}_3\, \underset{\sim}{v} = \bar{Q}\,. \tag{A9}$$

To reduce (A9) to two dimensions it is necessary to substitute for $\dfrac{\partial w}{\partial z}$ from (A7) which gives

$$h\,\rho\, \mathrm{div}_2\, \underset{\sim}{v} = \bar{Q}+h\, \mathrm{div}_2\, \rho \underset{\sim}{v}$$

or

$$\frac{\gamma}{\gamma-1}p\frac{\partial u_i}{\partial x_i} = \bar{Q}+h\frac{\partial \rho\, u_i}{\partial x_i} \qquad i = 1, 2\,. \tag{A10}$$

Both (A5) and (A10) show that the heat $\bar{Q}$ added to the flow is augmented by energy associated with the created fluid, but it is curious at first sight to note that this energy is less in (A10) where the created fluid (this term is convenient, regardless of how the extra fluid is in fact supposed to have been introduced) possesses kinetic energy, than in (A5) where the fluid is created with zero kinetic energy. The reason for this is if the velocity field is the same in the two cases, then there must be an extra body force in (A1) compared with (A8) of an amount $u_j \dfrac{\partial \rho\, u_i}{\partial x_i}$ and this force does work that more than compensates for the deficiency in kinetic energy.

A.2 *Relation between compressible and incompressible solutions.*

We know that the solution $\underline{v}_i$ which satisfies

$$\left.\begin{aligned} \rho_0\,(\underline{v}_i \operatorname{grad})\,\underline{v}_i + \operatorname{grad} p &= \underline{F}_i \\ \operatorname{div} \underline{v}_i &= G_i \end{aligned}\right\} \tag{A11}$$

is such that if $\underline{v} = \underline{v}_i \sqrt{\dfrac{\rho_0}{\rho}}$, then $\underline{v}$ satisfies

$$\left.\begin{aligned} \rho\,(\underline{v} \operatorname{grad})\,\underline{v} + \operatorname{grad} p &= \underline{f}_1 \\ \operatorname{div} \rho\underline{v} &= 0 \end{aligned}\right\} \tag{A12}$$

provided

$$\underline{f}_1 = \underline{F}_i + \rho_0\,\underline{v}_i\,G_i\,.$$

In the same way $\underline{v}$ satisfies (with m_c the mass created per unit volume per second)

$$\begin{aligned} \operatorname{div} \rho\underline{v} &= m_c \\ \rho\,(\underline{v} \operatorname{grad})\,\underline{v} + [\underline{v} \operatorname{div}(\rho\underline{v})] + \operatorname{grad} p &= \underline{f} \end{aligned} \tag{A13}$$

with

$$\underline{v} = \underline{v}_i \sqrt{\frac{\rho_0}{\rho}} \tag{A14}$$

provided

$$\underline{f} = \underline{F}_i + \rho\,(\underline{v} \operatorname{grad})\,\underline{v} + [\underline{v} \operatorname{div}(\rho\underline{v})] - \rho_0\,(v_i \operatorname{grad})\,v_i \tag{A15}$$

where the term in square brackets is present if (A1) applies but absent if (A8) applies. It is clear that outside the outer boundary of the arc where $\underline{f}$, $\underline{F}_i$ and $\operatorname{div}(\rho\underline{v})$ all vanish, then equation (A15) is satisfied as before.

To complete the conversion from incompressible to compressible flow, we must assume that the distribution of m_c is known and then the required relations can be established by considering continuity along a stream tube as before. This implies, for an element of stream tube of length ds and area A that

$$\frac{d}{ds}(\rho\, Au)\, ds = m_c\, A ds$$

or for a stream tube that passes right through the arc

$$\rho\, Au = \int_{-\infty} m_c\, A\, ds \tag{A16}$$

where m_c is zero except within the outer boundary of the arc. Since m_c and A are known (the latter from the incompressible solution) equation (A16) again gives a second equation for ρ and u which together with (A14) is sufficient to determine the velocity and density distributions in the compressible flow.

With mass being created in the arc, however, the stagnation point solutions in the incompressible flow can now be satisfied by a real flow. In this case some of the stream tubes originate within the arc from a point where the velocity vanishes (*see* Fig. 5), and for such stream tubes equation (A16) may be modified to

$$\rho\, Au = \int_0 m_c\, A\, ds \qquad \text{(A17)}$$

where $s = 0$ is the point which the stream tube originates. Equation (A17) together with (A14) again completely determines the flow field once the streamlines are known from the potential solution. In the wake with this type of flow the density is entirely dependent on m_c, and since pressure variations are assumed small this also determines the temperature. For a real arc heater in which axial flow and cross flow were combined, it would not be possible to control the distribution of m_c in the central regions of the column very directly, but the quasi-two-dimensional treatment suggested in this Appendix may still be useful if reliable measurements of the flow were available.

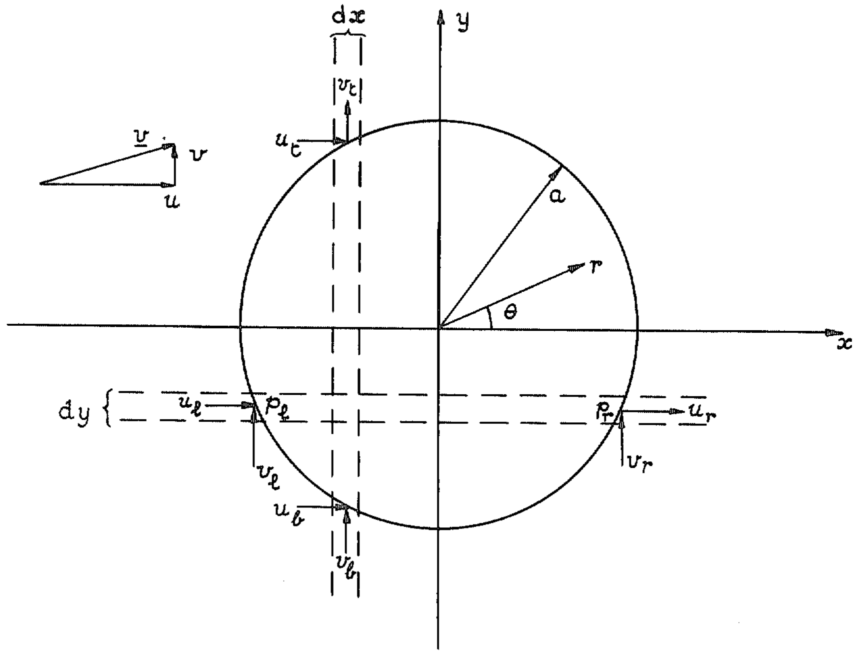

FIG. 1. Co-ordinate system.

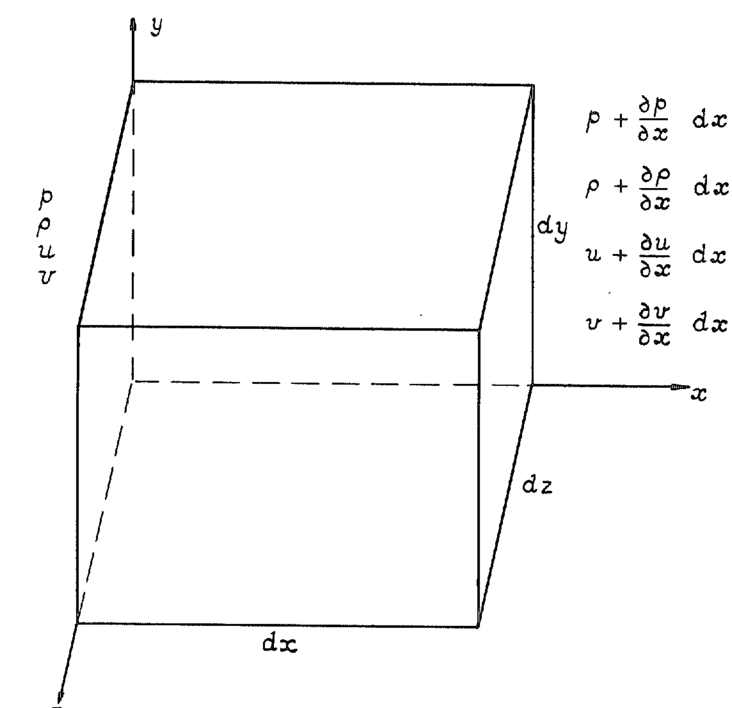

FIG. 2. Conservation diagram.

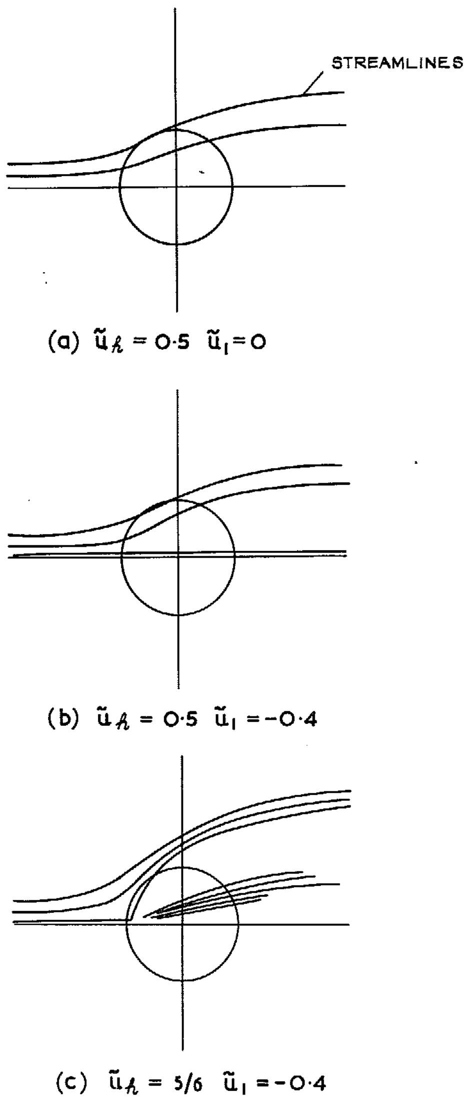

FIG. 3. Flow diagrams using ϕ_{2a} of Section 5.1.

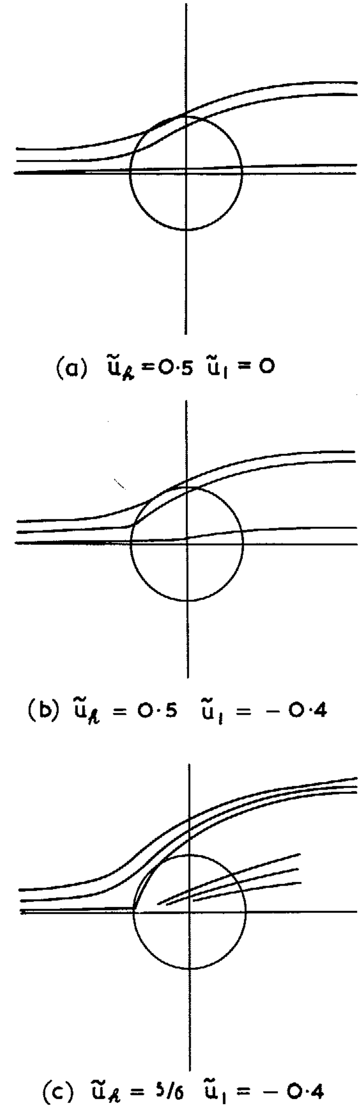

FIG. 4. Flow diagrams using ϕ_{2b} of Section 5.1.

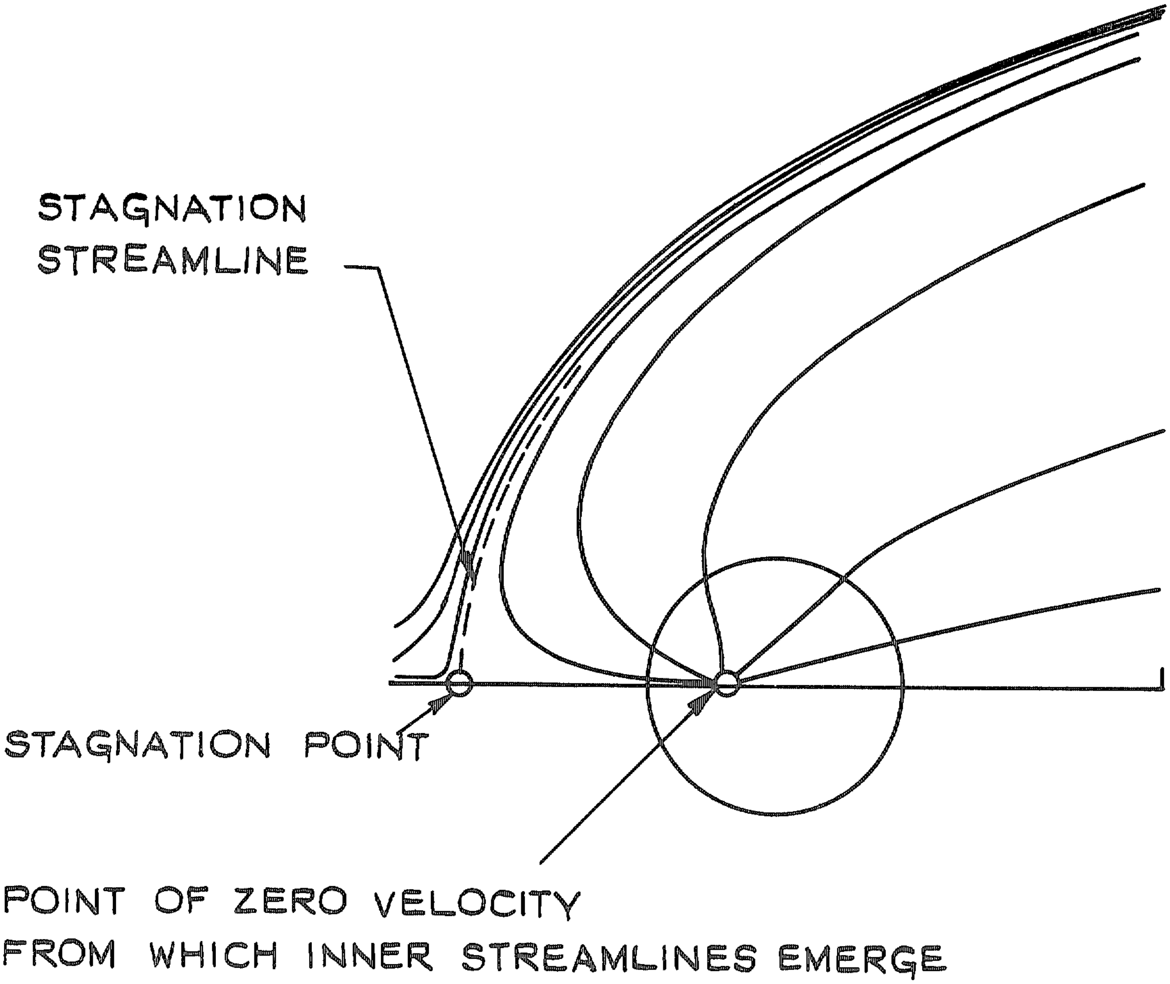

FIG. 5. Flow with stagnation point, ϕ_{2a} or ϕ_{2b} of Section 5.1.

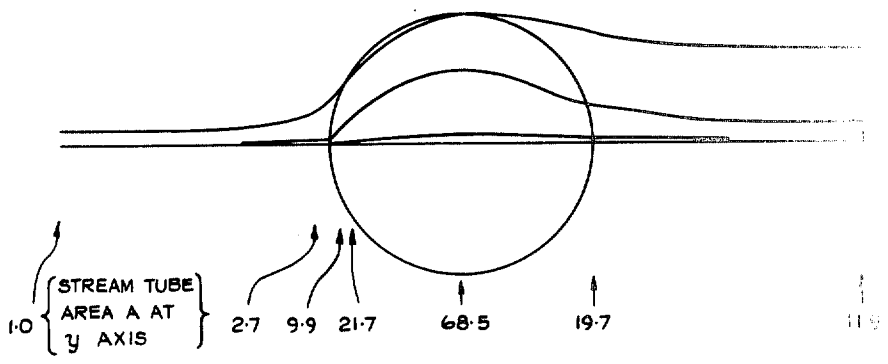

FIG. 6. Flow diagram for example of Section 5.2.: $\tilde{u}_1 = -0{\cdot}4$, $\tilde{u}_2 = -0{\cdot}5$.

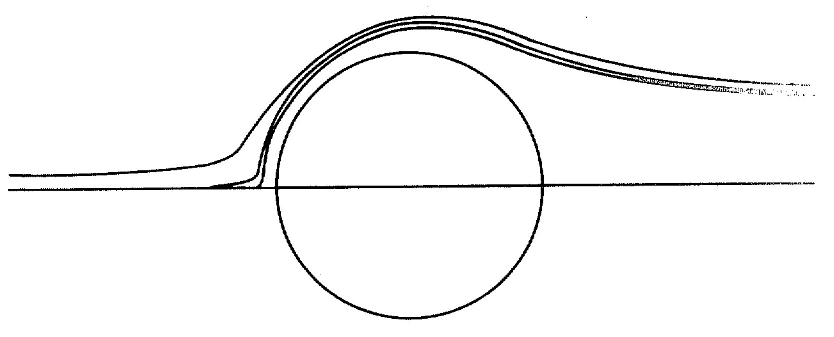

FIG. 7. Flow diagram for example of Section 5.2.: $\tilde{u}_1 = -0{\cdot}4$, $\tilde{u}_2 = -1{\cdot}0$.

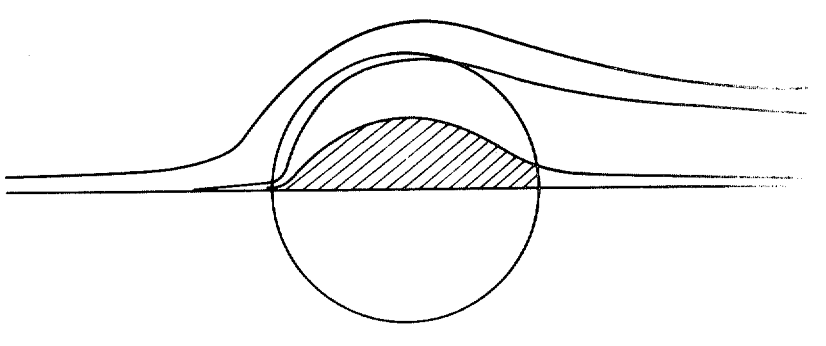

FIG. 8. Flow diagram for example of Section 5.2.: $\tilde{u}_1 = -0{\cdot}5$, $\tilde{u}_2 = -0{\cdot}65$.

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