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Summary.—(*a*) *Purpose of Note.*—To develop the theory of instability in both Euler and wrinkling modes under combined thrust and bending actions of sandwich struts or beams having an aeolotropic core material.

(*b*) *Range of Note.*—It is shown that the condition for instability under uniform end compression e and bending strain ψ (so that the compressive strains in the two skins are $e \pm \psi$) is

$$\psi^2 = (e_1 - e)\,(e_2 - e),$$

where e_1 and e_2 are critical strains in out-of-phase and in-phase modes of deformation respectively. Formulae for e_1 and e_2 are established, and the formula for e_2 agrees with the form previously derived by Williams, Leggett and Hopkins in R. & M. 1987[1], of which form it is a slight generalization. The variation of e_1 and e_2 with wave length of buckle and with the elastic properties of the core material is discussed.

(*c*) *Conclusions.*—For the great majority of practical core materials, true wrinkling of the skins over a short wavelength is a possible mode of failure and provided that the thickness of the core is at least 50 times the thickness of the skins, the critical strain e_w and the half-wave length π/α are expressible by the approximate formulae

$$e_w = 0\cdot 6\,(E_l\,E_d^{\,3}/E_s^{\,4})^{1/6}$$

and $$\pi/\alpha = 2\cdot 12 s\,(E_s^{\,4}/E_l\,E_d^{\,3})^{1/12} \text{ or } 1\cdot 64\, s/\sqrt{e_w},$$

where E_l = modulus of core material longitudinally,

E_d = modulus of core material transversely

and E_s = modulus of skin material.

Under uniform compression this instability occurs in the in-phase mode, the critical strain for the out-of-phase mode being slightly higher. In bending, or under combined loading, both in-phase and out-of-phase modes contribute to instability; but for all practical purposes the limiting condition is that the strain in the more heavily compressed skin shall be e_w.

The condition for modified Euler instability takes the usual form $1/P_c = 1/P_e + r$, where P_c is the critical load, P_e is the true Euler load and r is a shear constant, and this instability occurs, of course, over as long a wavelength as possible. Corresponding to the Euler in-phase mode there is an analogous out-of-phase mode, and in bending the two together represent a possible instability of Brazier type. Apart from this participation in instability over long wavelengths in bending, it is doubtful whether the long-wave out-of-phase mode would ever occur in a practical structure; but further examination is required to establish this conclusion.

(*d*) *Further Developments.*—Further examination is required of cases with aeolotropic cores and ratios of thicknesses of core to skin below 50.

* A.R.C. Report 9529 (April, 1946).

1. *Introduction.*—The purpose of the core in a sandwich structure is to stabilize the skins against local failure and to enable them to work together as a single beam, having a moment of section relatively much greater than that of the two skins separately. To fulfil these functions the core must restrain the skins from relative lateral movement and also limit relative shear movement between them ; a core which resisted shear only would permit out-of-phase buckling at a load determined purely by instability of the skins as struts ; whereas a core which resisted only lateral separation of the skins would permit in-phase buckling at the same load.

In Ref. 2 the incidence of these two types of instability was examined in respect of an artificial structure, in which in effect the core had an infinite stiffness longitudinally. This analysis indicated that the in-phase type of instability was of the modified Euler type over a long wavelength, the true Euler load being reduced by shear deformation of the core ; and that the out-of-phase mode should occur over a short wavelength and should depend principally on the modulus of the core material in the transverse direction. The formula derived for the critical strain in this out-of-phase mode included also an additive term proportional to the shear stiffness of the core ; owing to the inherent assumption that the core material was infinitely stiff longitudinally this effect was known to be overestimated.

In R. & M. 1987[1] the general problem of instability of a sandwich was examined in respect of a continuous core material having normal properties. The analysis-in-chief was carried through for an isotropic core, but the solution for an aeolotropic material was outlined in an Appendix. This analysis also identified the in-phase mode of instability over a long wavelength with the modified Euler mode ; but it indicated that the short wave-length mode should also occur in-phase, a conclusion previously drawn in Ref. 3 on the basis of a much less complete analysis.

If a sandwich is subjected not merely to uniform compression but to a compression e combined with a bending strain ψ, so that the skins are subjected to strains $e \pm \psi$, it is to be expected that the condition for instability should involve only ψ^2 and that it should reduce to $\psi = 0$ for any case in which $e = e_c$, where e_c is a critical strain under pure compression. When it is recognized that the imposition of a bending strain can introduce no characteristics of the structural properties which are not operative also under pure compression, it seems then inevitable that the condition for instability under combined loading should take the form $\psi^2 = A\,(e_1 - e)\,(e_2 - e)$, where A is a constant and e_1 and e_2 are two critical strains under pure compression. Moreover from the symmetric and antisymmetric nature of e and ψ respectively it is obvious that e_1 must refer to an out-of-phase mode and e_2 to an in-phase one. In fact it is proved later in this paper that all these presumptions are correct. On the other hand, for interaction between the modes of deformation represented by e_1 and e_2 the wavelengths of the buckles in the two modes must be the same, and to *each* value of e_2 for an assigned wavelength there corresponds a value of e_1 for the same wavelength. With a given ratio k of ψ to e there is then one pair of values of e_1 and e_2 with a common wavelength, which corresponds to the least possible value of e from the equation $k^2e^2 = A(e_1 - e)\,(e_2 - e)$; buckling would then occur over this wavelength and would include elements of both in-phase and out-of-phase deformation.

At the same time, if bending is entirely absent, k is zero and the least value of e reduces to e_1 or to e_2 simply, so that buckling under pure compression must occur, *either* in-phase *or* out-of-phase. It was shown in R. & M. 1987[1] that for true wrinkling of an isotropic core e_2 (in-phase) is slightly less than e_1 (out-of-phase) and that over longer wavelengths e_2 (modified Euler) should be considerably less than e_1 over the same wavelength. These conclusions for an isotropic core are confirmed below for an anisotropic core, except that a slight possibility remains that an extremely anisotropic core with relatively thick skins might wrinkle in the out-of-phase mode.

Although in this way in practically all cases of pure compression, the out-of-phase mode e_1 is secondary to e_2, in all cases of combined bending and direct load both e_1 and e_2 play essential parts. Over long wavelengths the combination of modified Euler with the associated mode of out-of-phase instability affords a type of Brazier instability in bending ; whilst the two values e_1 and e_2 for wrinkling are normally so close together that in practice instability may be said to occur when the more heavily loaded skin is stressed to a single critical value e_w.

In respect of the relative importance of the longitudinal and transverse moduli of the core material, the present analysis dealing throughout with an aeolotropic core shows that the value of e_w depends primarily on the value of the transverse modulus of the core. This analysis is developed in § 2, 3 and 4 below.

2. *A Method of Representing a State of Stress and Strain in an Aeolotropic Core.*—The elastic properties of the core material are assumed to be described by the relations :—

$$\left.\begin{aligned} e_{xx} &= s_{11}\overline{xx} + s_{12}\overline{yy} ,\\ e_{yy} &= s_{12}\overline{xx} + s_{22}\overline{yy} \\ \text{and} \quad e_{xy} &= s_{66}\,\overline{xy} , \end{aligned}\right\} \quad \ldots \quad (1)$$

and, provided that the deformation is small, $e_{xx} = u_x$, $e_{yy} = v_y$ and $e_{xy} = u_y + v_x$, where u and v are the displacements parallel to Ox and Oy respectively, and u_x stands for $(\partial u/\partial x)$ etc.

Assuming a stress function

$$\phi = (A_1 \cosh \lambda_1 \alpha y + A_2 \cosh \lambda_2 \alpha y + B_1 \sinh \lambda_1 \alpha y + B_2 \sinh \lambda_2 \alpha y) \sin \alpha x ,$$

it is sufficient to write only the first term and to expand where necessary by replacing $A_1 \cosh \lambda_1 \alpha y$ by $B_1 \sinh \lambda_1 \alpha y$, $A_1 \sinh \lambda_1 \alpha y$ by $B_1 \cosh \lambda_1 \alpha y$ and A_1 or B_1 with λ_1 by A_2 or B_2 with λ_2.

Then the stresses, strains and displacements are

$$\left.\begin{aligned} \overline{xx} &= \phi_{yy} = \lambda_1^2 \alpha^2 A_1 \cosh \lambda_1 \alpha y \sin \alpha x ,\\ \overline{yy} &= \phi_{xx} = -\alpha^2 A_1 \cosh \lambda_1 \alpha y \sin \alpha x ,\\ \overline{xy} &= -\phi_{xy} = -\lambda_1 \alpha^2 A_1 \sinh \lambda_1 \alpha y \cos \alpha x , \end{aligned}\right\} \quad \ldots \quad (2)$$

$$\left.\begin{aligned} u_x &= \alpha^2 (s_{11}\lambda_1^2 - s_{12})\, A_1 \cosh \lambda_1 \alpha y \sin \alpha x ,\\ v_y &= -\alpha^2 (s_{22} - s_{12}\lambda_1^2)\, A_1 \cosh \lambda_1 \alpha y \sin \alpha x \\ \text{and} \quad u_y + v_x &= -\alpha^2 s_{66} \lambda_1 A_1 \sinh \lambda_1 \alpha y \cos \alpha x , \end{aligned}\right\} \quad \ldots \quad (3).$$

and, by integration of the first two equations of (3)*,

$$\left.\begin{aligned} u &= -\alpha (s_{11}\lambda_1^2 - s_{12})\, A_1 \cosh \lambda_1 \alpha y \cos \alpha x \\ \text{and} \quad v &= -(\alpha/\lambda_1)(s_{22} - s_{12}\lambda_1^2)\, A_1 \sinh \lambda_1 \alpha y \sin \alpha x , \end{aligned}\right\} \quad \ldots \quad (4)$$

provided that to satisfy the third equation of (3)

$$\alpha^2 \lambda_1 (s_{11}\lambda_1^2 - s_{12}) + (\alpha^2/\lambda_1)(s_{22} - s_{12}\lambda_1^2) = \alpha^2 s_{66} \lambda_1 , \quad \ldots \quad (5)$$

or

$$\lambda_1^4 - \{(s_{66} + 2s_{12})/s_{11}\}\, \lambda_1^2 + (s_{22}/s_{11}) = 0 . \quad \ldots \quad (5a)$$

Clearly λ_1 and λ_2 must be the roots of (5a), so that

$$\left.\begin{aligned} \lambda_1 \lambda_2 &= (s_{22}/s_{11})^{1/2} \\ \text{and} \quad (\lambda_1^2 + \lambda_2^2)/\lambda_1\lambda_2 &= (s_{66} + 2s_{12})/(s_{11}s_{22})^{1/2} , \end{aligned}\right\} \quad \ldots \quad (6)$$

and we may assume that $\lambda_1 > \lambda_2$. It is convenient to write $E = 1/(s_{11}s_{22})^{1/2}$; and $\sigma = -s_{12}/(s_{11}s_{22})^{1/2}$; then formulae (4) may be rewritten in the forms

$$\left.\begin{aligned} Eu &= -\alpha (\sigma + \lambda_1/\lambda_2)\, A_1 \cosh \lambda_1 \alpha y \cos \alpha x \\ \text{and} \quad Ev &= -\alpha (\lambda_2 + \sigma\lambda_1)\, A_1 \sinh \lambda_1 \alpha y \sin \alpha x , \end{aligned}\right\} \quad \ldots \quad (7)$$

* Omitting constants of integration, which are unnecessary.

and from the second of (6)

$$E/G_d = s_{66}/(s_{11}s_{22})^{1/2} = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1\lambda_2} + 2\sigma = 2(1 + \sigma) + \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1\lambda_2}. \quad \ldots \quad \ldots \quad (8)$$

To summarize: the elastic constants of the core material are represented by the three ratios λ_1, λ_2 and σ, and

(*a*) the ratio of the longitudinal modulus E_l to the transverse modulus E_d is $E_l/E_d = s_{22}/s_{11} = \lambda_1^2\lambda_2^2$; the "standard" E is the geometrical mean $(E_lE_d)^{1/2}$:

(*b*) σ/E is the transverse contraction due to unit tensile stress longitudinally or *vice versa*; the two Poisson's ratios are $\sigma_{12} = \sigma E_l/E$ and $\sigma_{21} = \sigma E_d/E$, so that $\sigma = (\sigma_{12}\sigma_{21})^{1/2}$:

(*c*) the excess of E/G_d over $2(1 + \sigma)$ is $(\lambda_1 - \lambda_2)^2/\lambda_1\lambda_2$ and defines the ratio λ_1/λ_2 (but *see* note below on the practical aspect of this relation).

3. *Values of the Coefficients* λ_1 *and* λ_2.—If $E/G_d \equiv 2\,(1 + \sigma)$, $\lambda_1 \equiv \lambda_2$ and the form $A_1 \cosh \lambda_1\alpha y + A_2 \cosh \lambda_2\alpha y$ reduces to one term only. However, writing $\lambda_1 = \lambda + \delta\lambda$ and $\lambda_2 = \lambda - \delta\lambda$, where $\delta\lambda$ is small, the form may be written $(A_1 + A_2) \cosh \lambda\alpha y + (A_1 - A_2)\delta\lambda\alpha y \sinh \lambda\alpha y$; and, by assuming that $A_1 - A_2$ tends to infinity as $\delta\lambda$ decreases to zero, the form $P \cosh \lambda\alpha y + Q\lambda\alpha y \sinh \lambda\alpha y$ is derived. This form could be discussed separately (as in R. & M. 1987[1]), but in fact the limiting process may be performed at any stage so that independent investigation is not strictly necessary.

If $E/G_d < 2(1 + \sigma)$, λ_1 and λ_2 are complex. Since $\lambda_1\lambda_2$ is real, λ_1 and λ_2 must be conjugate, so that $\lambda_1 = \lambda + i\mu$ and $\lambda_2 = \lambda - i\mu$. Then $E_l/E_d = \lambda_1^2\lambda_2^2 = (\lambda^2 + \mu^2)^2$ and $E/G_d - 2(1 + \sigma) = (\lambda_1 - \lambda_2)^2/\lambda_1\lambda_2 = -\,4\mu^2/(\lambda^2 + \mu^2)$. Thus only in exceptional cases can $\mu > \lambda$. If $\mu \to 0$, λ_1 and λ_2 tend to equality; this is a convenient approach to the special case when $E = 2G_d(1 + \sigma)$.

However, unless E/G_d is known to be *identical* with $2(1 + \sigma)$, experimental data can establish the equality $\lambda_1 = \lambda_2$ only to a limited accuracy; for instance, if $E/G_d = 2(1 + \sigma)$ with a possible error of $\pm\,0{\cdot}01$, then $\lambda_1/\lambda_2 < 1{\cdot}1$. Thus 1 per cent. error in the value of E or G_d or about 2 per cent. error in the value of σ, permits 10 per cent. variation of the ratio λ_1/λ_2; thus in practice no real restriction of applicability results from exclusion of the case $\lambda_1 = \lambda_2$.

4. *Conditions for Equilibrium of the Skins*.—The values at the surfaces of the core at $y = \pm\, d/2$ of $\overline{yy}$ will be denoted by p and p', of $\overline{xy}$ by q and q', of u by U and U', and of v by V and V'. The direct thrusts in the skins of thickness s will be denoted by P and P', the shear forces by F and F', and the bending moments by M and M'. The half wavelength of the deformation along the length of the sandwich is of course π/α.

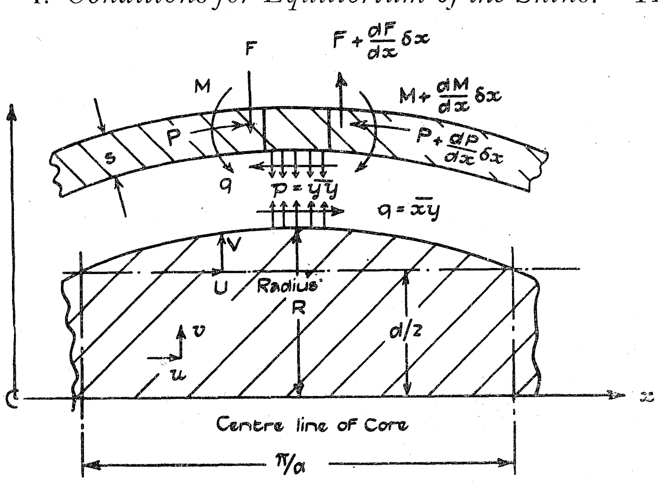

FIG. 1.

Then, with the sign conventions indicated in Fig. 1, the equations of equilibrium of the skin at $y = d/2$ are, by resolving forces normal to the skin,

$$p = \frac{dF}{dx} + \frac{P}{R}, \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

by resolving forces along the skin

$$\frac{dP}{dx} + q = \frac{F}{R} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

and by taking moments

$$\frac{dM}{dx} + \frac{qs}{2} = F \,. \qquad \ldots \qquad (11)$$

Moreover, provided that V is small, $1/R = -\,d^2V/dx^2$ and $M = -\frac{1}{12} E_s's^3 . \, d^2V/dx^2$, where $E_s' = E_s/(1 - \sigma_s^2)$, E_s is the Young's modulus of the skin material and σ_s its Poisson's ratio*. Initially, when $V = 0$, $P = P_0$(const) and $F = 0$; therefore F/R and the variable part of P/R are of order V^2 and may be neglected. Then by elimination of F, formulae (9) and (11) give

$$p + \frac{1}{12} E_s's^3 \frac{d^4V}{dx^4} - \frac{s}{2}\frac{dq}{dx} + P_0 \frac{d^2V}{dx^2} = 0 \qquad \ldots \qquad (12)$$

(which corresponds to equation (6) of R. & M. 1987[1]).

Moreover

$$\frac{dU}{dx} = -\frac{P}{E_s s} - \frac{s}{2R} = \frac{s}{2}\frac{d^2V}{dx^2} - \frac{P}{E_s s} ,$$

or

$$\frac{d^2U}{dx^2} = \frac{s}{2}\frac{d^3V}{dx^3} - \frac{1}{E_s s}\frac{dP}{dx} = \frac{s}{2}\frac{d^3V}{dx^3} + \frac{q}{E_s s} , \qquad \ldots \qquad (13)$$

by using (10). (Equation (13) corresponds to equation (9), and the first part of equation (10) of R. & M. 1987[1].) At the opposite face ($y = -\,d/2$), p is replaced by p', q by $-\,q'$, U by U', V by $-\,V'$ and P_0 by P_0', so that the formulae analogous to (12) and (13) are

$$p' - \frac{1}{12} E_s's^3 \frac{d^4V'}{dx^4} + \frac{s}{2}\frac{dq'}{dx} - P_0' \frac{d^2V'}{dx^2} = 0 \qquad \ldots \qquad (14)$$

and

$$\frac{d^2U'}{dx^2} = -\frac{s}{2}\frac{d^3V'}{dx^3} - \frac{q'}{E_s s} \qquad \ldots \qquad (15)$$

By adding and subtracting (12) and (14) and (13) and (15), the four relations become

$$p + p' + \frac{1}{12} E_s's^3 \frac{d^4}{dx^4}(V - V') - \frac{s}{2}\frac{d}{dx}(q - q') + \frac{(P_0 + P_0')}{2}\frac{d^2}{dx^2}(V - V') + \frac{(P_0 - P_0')}{2}\frac{d^2}{dx^2}(V + V') = 0 \,, \qquad \ldots \qquad (16)$$

$$p - p' + \frac{1}{12} E_s's^3 \frac{d^4}{dx^4}(V + V') - \frac{s}{2}\frac{d}{dx}(q + q') + \frac{(P_0 + P_0')}{2}\frac{d^2}{dx^2}(V + V') + \frac{(P_0 - P_0')}{2}\frac{d^2}{dx^2}(V - V') = 0 \,, \qquad \ldots \qquad (17)$$

$$\frac{d^2}{dx^2}(U + U') = \frac{s}{2}\frac{d^3}{dx^3}(V - V') + \frac{1}{E_s s}(q - q') \qquad \ldots \qquad (18)$$

and

$$\frac{d^2}{dx^2}(U - U') = \frac{s}{2}\frac{d^3}{dx^3}(V + V') + \frac{1}{E_s s}(q + q') \,. \qquad \ldots \qquad (19)$$

* The value 0·3 is assumed for σ_s, where necessary to save complication later.

Then from (7)

$$\left.\begin{aligned} E\,(U+U') &= -\,2\alpha\{(\sigma+\lambda_1/\lambda_2)\,A_1\cosh(\lambda_1\alpha d/2)+(\sigma+\lambda_2/\lambda_1)\,A_2\cosh(\lambda_2\alpha d/2)\}\cos\alpha x\,;\\ E\,(U-U') &= -\,2\alpha\{(\sigma+\lambda_1/\lambda_2)\,B_1\sinh(\lambda_1\alpha d/2)+(\sigma+\lambda_2/\lambda_1)\,B_2\sinh(\lambda_2\alpha d/2)\}\cos\alpha x\,,\\ E\,(V-V') &= -\,2\alpha\{(\lambda_2+\sigma\lambda_1)\,A_1\sinh(\lambda_1\alpha d/2)+(\lambda_1+\sigma\lambda_2)\,A_2\sinh(\lambda_2\alpha d/2)\}\sin\alpha x\\ \text{and}\;\; E\,(V+V') &= -\,2\alpha\{(\lambda_2+\sigma\lambda_1)\,B_1\cosh(\lambda_1\alpha d/2)+(\lambda_1+\sigma\lambda_2)\,B_2\cosh(\lambda_2\alpha d/2)\}\sin\alpha x\,; \end{aligned}\right\}\quad (20)$$

and from (2)

$$\left.\begin{aligned} p+p' &= -\,2\alpha^2\{A_1\cosh(\lambda_1\alpha d/2)+A_2\cosh(\lambda_2\alpha d/2)\}\sin\alpha x\,,\\ p-p' &= -\,2\alpha^2\{B_1\sinh(\lambda_1\alpha d/2)+B_2\sinh(\lambda_2\alpha d/2)\}\sin\alpha x\,,\\ q-q' &= -\,2\alpha^2\{\lambda_1A_1\sinh(\lambda_1\alpha d/2)+\lambda_2A_2\sinh(\lambda_2\alpha d/2)\}\cos\alpha x\\ \text{and}\qquad q+q' &= -\,2\alpha^2\{\lambda_1B_1\cosh(\lambda_1\alpha d/2)+\lambda_2B_2\cosh(\lambda_2\alpha d/2)\}\cos\alpha x\,. \end{aligned}\right\}\quad \ldots\quad \ldots\quad (21)$$

Substituting for $U+U'$ etc. from (20) and (21) in (18) and for brevity writing C_1 for $\cosh(\lambda_1\alpha d/2)$, S_2 for $\sinh(\lambda_1\alpha d/2)$, C_2 for $\cosh(\lambda_2\alpha d/2)$ and S_2 for $\sinh(\lambda_2\alpha d/2)$,

$$\alpha^3\{(\sigma+\lambda_1/\lambda_2)\,A_1C_1+(\sigma+\lambda_2/\lambda_1)\,A_2C_2\}=\tfrac{1}{2}\alpha^4s\{(\lambda_2+\sigma\lambda_1)\,A_1S_1+(\lambda_1+\sigma\lambda_2)\,A_2S_2\}$$
$$-\,(E/E_ss)\,\alpha^2\,(\lambda_1A_1S_1+\lambda_2A_2S_2)\,,\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad (22)$$

or $$\{\alpha s\,(\sigma+\lambda_1/\lambda_2)\,C_1-\tfrac{1}{2}\,(\alpha s)^2\,(\lambda_2+\sigma\lambda_1)\,S_1+(E/E_s)\,\lambda_1S_1\}A_1$$
$$+\,\{\alpha s\,(\sigma+\lambda_2/\lambda_1)\,C_2-\tfrac{1}{2}\,(\alpha s)^2\,(\lambda_1+\sigma\lambda_2)\,S_2+(E/E_s)\,\lambda_2S_2\}A_2=0\,,\quad \ldots\quad \ldots\quad (22a)$$

and (19) leads to an exactly similar relation with B for A, S for C and C for S.

Substituting for $p+p'$ etc, from (20) and (21) in (16),

$$-\,\alpha^2\,(A_1C_1+A_2C_2)-\tfrac{1}{12}\,(E'_ss^3/E)\alpha^5\,\{(\lambda_2+\sigma\lambda_1)\,A_1S_1+(\lambda_1+\sigma\lambda_2)A_2S_2\}$$
$$-\,\tfrac{1}{2}\alpha^3s\,(\lambda_1A_1S_1+\lambda_2A_2S_2)+\{(P_0+P_0')/2E\}\alpha^3\,\{(\lambda_2+\sigma\lambda_1)\,A_1S_1+(\lambda_1+\sigma\lambda_2)\,A_2S_2\}$$
$$+\,\{(P_0-P_0')/2E\}\alpha^3\{(\lambda_2+\sigma\lambda_1)\,B_1C_1+(\lambda_1+\sigma\lambda_2)\,B_2C_2\}=0\,;\quad \ldots\quad \ldots\quad \ldots\quad (23)$$

and writing $(P_0+P_0')/2E_ss=e$ and $(P_0-P_0')/2E_ss=\psi$, this becomes

$$A_1C_1+A_2C_2+\tfrac{1}{12}\,(E_s'/E)\,(\alpha s)^3\,\{(\lambda_2+\sigma\lambda_1)\,A_1S_1+(\lambda_1+\sigma\lambda_2)\,A_2S_2\}$$
$$+\,\tfrac{1}{2}\,(\alpha s)\,(\lambda_1A_1S_1+\lambda_2A_2S_2)=(E_s/E)\,(\alpha s)\,[e\{(\lambda_2+\sigma\lambda_1)\,A_1S_1+(\lambda_1+\sigma\lambda_2)\,A_2S_2\}$$
$$+\,\psi\{(\lambda_2+\sigma\lambda_1)\,B_1C_1+(\lambda_1+\sigma\lambda_2)\,B_2C_2\}]\,,\quad \ldots\quad \ldots\quad \ldots\quad (23a)$$

and similarly from equation (17) with B for A, A for B, C for S and S for C.

Multiplying (23a) by the coefficient of A_2 in (22a) and eliminating A_2 gives

$$(E/E_s)\,(\lambda_1T_1-\lambda_2T_2)+(\alpha s)\{(\lambda_1^2-\lambda_2^2)/\lambda_1\lambda_2\}+(\alpha s)^2\{(\lambda_1+\sigma\lambda_2)\,T_2-(\lambda_2+\sigma\lambda_1)\,T_1\}$$
$$+\,0\cdot3^4\,(\alpha s)^3\,(\lambda_1^2-\lambda_2^2)\,T_1T_2+\tfrac{1}{12}\,(E_s'/E)\,(\alpha s)^4\{(\sigma+\lambda_1/\lambda_2)^2\,\lambda_2T_2-(\sigma+\lambda_2/\lambda_1)^2\,\lambda_1T_1\}$$
$$=e\,(\alpha s)\,[(\lambda_1^2-\lambda_2^2)\,T_1T_2+(E_s/E)\,(\alpha s)\{(\sigma+\lambda_1/\lambda_2)^2\,\lambda_2T_2-(\sigma+\lambda_2/\lambda_1)^2\,\lambda_1T_1\}]$$
$$+\,\psi\,(\alpha s)\,[(\lambda_1^2-\lambda_2^2)+(E_s/E)\,(\alpha s)\{(\sigma+\lambda_1\lambda_2)^2\,\lambda_2T_1-(\sigma+\lambda_2/\lambda_1)^2\,\lambda_1T_2\}]\times K\,,\quad \ldots\quad (24)$$

where T_1 stands for $\tanh(\lambda_1\alpha d/2)$, T_2 for $\tanh(\lambda_2\alpha d/2)$ and

$$K=\frac{\{(\alpha s)\,(\sigma+\lambda_2/\lambda_1)\,C_2-\tfrac{1}{2}\,(\alpha s)^2\,(\lambda_1+\sigma\lambda_2)\,S_2+(E/E_s)\,\lambda_2S_2\}}{\{(\alpha s)\,(\sigma+\lambda_2/\lambda_1)\,S_2-\tfrac{1}{2}\,(\alpha s)^2\,(\lambda_1+\sigma\lambda_2)\,C_2+(E/E_s)\,\lambda_2C_2\}}\times\left(\frac{B_1}{A_1}\right).$$

Equation (24) is of the form $Me + N\psi K = Me_1$ and the analogous equation with A and B interchanged is $Ne + M\psi/K = Ne_2$, so that the two equations may be combined in the form

$$\psi^2 = (e_1 - e)\,(e_2 - e)\,, \qquad \ldots \qquad (25)$$

where

$$e_1 = \frac{(\alpha s)^2}{12\,(1-\sigma_s^2)} + \frac{P(E/E_s) + \alpha s + Q\,(\alpha s)^2 + \frac{1}{4}\lambda_1\lambda_2 T_1 T_2\,(\alpha s)^3}{(\alpha s)\{R\,(\alpha s) + \lambda_1\lambda_2 T_1 T_2\,(E/E_s)\}} \times \frac{E}{E_s} \qquad \ldots \qquad (26)$$

and

$$e_2 = \frac{(\alpha s)^2}{12\,(1-\sigma_s^2)} + \frac{P'\,(E/E_s) + \alpha s + Q'\,(\alpha s)^2 + \frac{1}{4}\,(\lambda_1\lambda_2/T_1T_2)\,(\alpha s)^3}{(\alpha s)\{R'\,(\alpha s) + (\lambda_1\lambda_2/T_1T_2)\,(E/E_s)\}} \times \frac{E}{E_s}\,, \qquad (27)$$

and

$$\left.\begin{aligned}
P &= \lambda_1\lambda_2\,(\lambda_1 T_1 - \lambda_2 T_2)/(\lambda_1^2 - \lambda_2^2)\,,\\
P' &= \lambda_1\lambda_2\{(\lambda_1/T_1) - (\lambda_2/T_2)\}/(\lambda_1^2 - \lambda_2^2)\,,\\
Q &= \lambda_1\lambda_2\{(\lambda_1 + \sigma\lambda_2)\,T_2 - (\lambda_2 + \sigma\lambda_1)\,T_1\}/(\lambda_1^2 - \lambda_2^2)\,,\\
Q' &= \lambda_1\lambda_2\{(\lambda_1 + \sigma\lambda_2)/T_2 - (\lambda_2 + \sigma\lambda_1)/T_1\}/(\lambda_1^2 - \lambda_2^2)\,,\\
R &= \{(\lambda_1 + \sigma\lambda_2)^2\,\lambda_1 T_2 - (\lambda_2 + \sigma\lambda_1)^2\,\lambda_2 T_1\}/(\lambda_1^2 - \lambda_2^2)\\
\text{and}\quad R' &= \{(\lambda_1 + \sigma\lambda_2)^2\,(\lambda_1/T_2) - (\lambda_2 + \sigma\lambda_1)^2\,(\lambda_2/T_1)\}/(\lambda_1^2 - \lambda_2^2)\,.
\end{aligned}\right\} \qquad \ldots \qquad (28)$$

e_1 and e_2 are conjugate in the sense that e_2 has $1/T_1$ and $1/T_2$ where e_1 has T_1 and T_2 respectively.

By writing $\lambda_1 = \lambda + \mu$ and $\lambda_2 = \lambda - \mu$, formulae (28) may be put in the forms

$$\left.\begin{aligned}
P,\,P' &= (\lambda^2 - \mu^2)\{(\sinh\lambda\alpha d/2\lambda) \pm (\sinh\mu\alpha d/2\mu)\}/(\cosh\lambda\alpha d \pm \cosh\mu\alpha d)\\
Q,\,Q' &= (\lambda^2 - \mu^2)\{(1-\sigma)\,(\sinh\lambda\alpha d/2\lambda) \mp (1+\sigma)\,(\sinh\mu\alpha d/2\mu)\}/(\cosh\lambda\alpha d \pm \cosh\mu\alpha d)\\
\text{and } R,\,R' &= [\{3-\sigma)\,(1+\sigma)\lambda^2 + (1-\sigma)^2\,\mu^2\}(\sinh\lambda\alpha d/2\lambda)\\
&\quad \mp\{(1+\sigma)^2\,\lambda^2 + (3+\sigma)\,(1-\sigma)\,\mu^2\}(\sinh\mu\alpha d/2\mu)]/[\cosh\lambda\alpha d \pm \cosh\mu\alpha d]
\end{aligned}\right\} \qquad (29)$$

(the alternative signs in numerators and denominators being paired), and

$$T_1 T_2 = (\cosh\lambda\alpha d - \cosh\mu\alpha d)/(\cosh\lambda\alpha d + \cosh\mu\alpha d)\,.$$

If $E/G_d < 2\,(1+\sigma)$, $\lambda_{1,2} = \lambda \pm i\mu$ (§ 3) and the forms for P, Q etc. follow from (29) by substituting $i\mu$ for μ; when μ^2 becomes $-\mu^2$, $(\sinh\mu\alpha d/2\mu)$ becomes $(\sin\mu\alpha d/2\mu)$ and $\cosh\mu\alpha d$ becomes $\cos\mu\alpha d$.

If $\mu = 0$, so that $\lambda_1 = \lambda_2 = \lambda$, the formulae (29) reduce to

$$\left.\begin{aligned}
P,\,P' &= \tfrac{1}{2}\lambda\,(\sinh\lambda\alpha d \pm \lambda\alpha d)/(\cosh\lambda\alpha d \pm 1)\\
Q,\,Q' &= \tfrac{1}{2}\lambda\,\{(1-\sigma)\sinh\lambda\alpha d \mp (1+\sigma)\,\lambda\alpha d\}/(\cosh\lambda\alpha d \pm 1)\\
\text{and } R,\,R' &= \tfrac{1}{2}\lambda\,(1+\sigma)\{(3-\sigma)\sinh\lambda\alpha d \mp (1+\sigma)\,\lambda\alpha d\}/(\cosh\lambda\alpha d \pm 1).
\end{aligned}\right\} \qquad \ldots \qquad (30)$$

When $\lambda = 1$ the formula (27) with the values of P', Q' and R' given by (30) corresponds to formulae (13) and (14) of R. & M. 1987[1].

5. *Wrinkling in Compression, Bending or Under Combined Loading.*—Presuming for the present that, with the appropriate value of α to give the least critical strain, αd is large in the sense that $T_1 = T_2 = 1$ with negligible error, we may write $e_1 = e_2 = e_w$, where e_w is given by (26) or (27) with $P = P' = \lambda_1\lambda_2/(\lambda_1 + \lambda_2)$, $Q = Q' = (1-\sigma)\,\lambda_1\lambda_2/(\lambda_1 + \lambda_2)$ and $R = R' = \lambda_1 + \lambda_2 - (1-\sigma)^2\,\lambda_1\lambda_2/(\lambda_1 + \lambda_2)$ or $\{(E/G_d) + 1 - \sigma^2\}\lambda_1\lambda_2/(\lambda_1 + \lambda_2)$.

Then from (25), $\pm\,\psi = e_w - e$ expresses the condition for wrinkling, that is that the stress in the more heavily loaded skin should equal $E_s e_w$, where

$$e_w = \frac{(\alpha s)^2}{12(1-\sigma_s^2)} + \frac{(E/E_s) + \{(\lambda_1+\lambda_2)/\lambda_1\lambda_2\}(\alpha s) + (1-\sigma)(\alpha s)^2 + \frac{1}{4}(\lambda_1+\lambda_2)(\alpha s)^3}{(\alpha s)\,[\{(E/G_d) + 1 - \sigma^2\}(\alpha s) + (\lambda_1+\lambda_2)(E/E_s)]} \times \frac{E}{E_s}. \quad (31)$$

Clearly, if e_w is to be a practical limit within the range of elasticity of the skins, αs must be small ($\not> 0\cdot25$ say) and $(\alpha s)^2$ and $(\alpha s)^3$ may be negligible in comparison. In the second term the coefficients of αs in the numerator and of $(\alpha s)^2$ in the denominator are of the same order, so that for the same reason $E/E_s\alpha s$ must be small ($\not> 0\cdot01$ say). Therefore

$$e_w \doteqdot \frac{(\alpha s)^2}{12\,(1-\sigma_s^2)} + \frac{(E/E_s)\,(\lambda_1+\lambda_2)/\lambda_1\lambda_2}{\{(E/G_d) + 1 - \sigma^2\}(\alpha s)} \quad \ldots \quad (32)$$

and the minimum value

$$e_w \doteqdot \frac{1}{4\,(1-\sigma_s^2)}\left[\frac{6\,(1-\sigma_s^2)\,(E/E_s)\,(\lambda_1+\lambda_2)}{\lambda_1\lambda_2\{(E/G_d) + 1 - \sigma^2\}}\right]^{2/3} \quad \ldots \quad (33)$$

is given by

$$\alpha s = [6\,(1-\sigma_s^2)\,(E/E_s)\,(\lambda_1+\lambda_2)/\lambda_1\lambda_2\{(E/G_d) + 1 - \sigma^2\}]^{1/3}\,. \quad \ldots \quad (34)$$

Since $(\lambda_1 - \lambda_2)^2/\lambda_1\lambda_2 = (E/G_d) - 2\,(1+\sigma)$, $(\lambda_1+\lambda_2)^2/\lambda_1\lambda_2 = (E/G_d) + 2\,(1-\sigma)$, and since $\lambda_1\lambda_2 = (E_l/E_d)^{1/2}$, $(\lambda_1+\lambda_2)/\lambda_1\lambda_2 = \{(E/G_d) + 2\,(1-\sigma)\}^{1/2}\,(E_d/E_l)^{1/4}$. Substituting in (33),

$$e_w \doteqdot \frac{1}{4\,(1-\sigma_s^2)}\left[\frac{6\,(1-\sigma_s^2)\{(E/G_d) + 2\,(1-\sigma)\}^{1/2}}{\{(E/G_d) + 1 - \sigma^2\}} \times \frac{(E_l/E_d^3)^{1/4}}{E_s}\right]^{2/3}. \quad \ldots \quad (33a)$$

If $E/G_d = 2\,(1+\sigma)$, so that $\lambda_1 = \lambda_2$,

$$e_w \doteqdot \frac{1}{4\,(1-\sigma_s^2)}\left\{\frac{12\,(1-\sigma_s^2)}{(3-\sigma)\,(1+\sigma)}\right\}^{2/3}\left(\frac{E_l^{1/6}E_d^{1/2}}{E_s^{2/3}}\right) \quad \ldots \quad (33b)$$

$$\doteqdot (0\cdot651 - 0\cdot22\sigma)\,(E_l^{1/6}E_d^{1/2}/E_s^{2/3})\,. \quad \ldots \quad (33c)$$

When $\sigma_s = 0\cdot3$ and $0 < \sigma < 0\cdot3$, 10 per cent. increase of E/G_d over $2\,(1+\sigma)$ causes about 3 per cent. decrease of the numerical factor and *vice versa*. The accuracy of (32) or even of (31) is not high enough to justify great precision, so that this first approximation to e_w may be taken simply as

$$e_w = 0\cdot6\,(E_l E_d^3/E_s^4)^{1/6}\,, \quad \ldots \quad (33d)$$

corresponding to

$$\alpha s = 1\cdot48\,(E_l E_d^3/E_s^4)^{1/12}\,. \quad \ldots \quad (34d)$$

By reference to formula (33) it may be shown that variation of the ratio λ_1/λ_2 over the range 1 to $2\cdot5$ causes less than 10 per cent. variation of the constant given as $0\cdot6$ in formula (33d), increase of λ_1/λ_2 causing a decrease of this factor. Over this range of λ_1/λ_2 the effect of variation of σ is adequately represented by formula (33c). The value of e_w, defined by (33d), will be a practical limit only if it be less than $0\cdot006$ (say); therefore $(E_l E_d^3/E_s^4)^{1/6}$ must be less than $0\cdot01$, and $\alpha s < 0\cdot15$. On the other hand a value of e_w below (say) $0\cdot003$ is unlikely to represent an efficient structure, so that αs can scarcely be less than $0\cdot10$. For values of λ_1 and λ_2 not much less than unity a value of αd about 6 is sufficient to justify the presumption that $T_1 = T_2 = 1$ to 1 per cent. accuracy, so that formulae (33) and (34) are applicable to values of d/s exceeding (say) 50. For lower values of this ratio, the forms (29) or (30) for P, Q etc. must be used.

6. *Wrinkling of Thin Sandwiches.*—The minimum value of about 50 for the ratio of thickness of core to thickness of skin necessary to justify the conclusions of § 5 is unfortunately rather high, and in many practical cases the assumption that $T_1 = T_2 = 1$ with negligible error is not justified. When αd is not large each of the several terms in formulae (26) and (27) influences

the values of e_1 and e_2, and the minimum values of e_1 and e_2 depend upon all the elastic constants and on the ratio s/d in very complicated ways. However for true wrinkling characterized by moderately high values of αs (say about $0 \cdot 10$), the forms

$$e_1 = K_1(E_l E_d^3/E_s^4)^{1/6} \text{ and } e_2 = K_2(E_l E_d^3/E_s^4)^{1/6} \qquad \text{(35)}$$

are still justifiable in the sense that K_1 and K_2 are functions principally of the ratio s/d and depend only to a small extent on the values of the elastic constants. In these cases e_2 is normally less than e_1, so that in pure compression wrinkling should occur in the in-phase mode in accordance with the conclusions of R. & M. 1987[1]. Although the cases in which the minimum value of e_1 is less than that of e_2 may be quite abnormal, the difference between the two minima is usually very small, and, in any case, for discussion of wrinkling under bending or under bending and compression the values of both e_1 and e_2 are required (*cf.* formula (25)).

For practical application the chief general conclusion which may tentatively be drawn with respect to wrinkling is that the critical stress is proportional to $(E_l E_d^3 E_s^2)^{1/6}$, the constant of proportionality being a function of s/d but almost independent of the elastic constants. The validity of this conclusion needs to be examined further, but the form $(E_l E_d^3 E_s^2)^{1/6}$ is emphasized here because it indicates the relative importance of the three moduli. It is important to appreciate that deformation in the in-phase mode of wrinkling, which might appear to depend principally on the longitudinal modulus E_l, is in fact no less dependent on the cross modulus E_d than the out-of-phase mode. As α decreases, that is, as the wavelength of buckle grows longer, the variation of K_2 in formula (35) with the elastic moduli increases and the form (35) becomes less appropriate, until, when α is very small, the instability is the modified Euler type, and its incidence is determined almost entirely by the values of E_l and G_d, whilst variation of E_d *per se* has practically no effect. On the other hand for low values of α the value of e_1 is determined almost entirely by the value of E_d and is practically independent of the value of E_l.

The values of e_1 and e_2 at long wavelengths of buckle may thus appear virtually independent; but in fact G_d the shear modulus is normally closely associated with the mean modulus $E = (E_l E_d)^{1/2}$. When this association is taken into account, it seems that over this range also e_2 is normally less than e_1, so that out-of-phase buckling may occur only in exceptional cases. On the other hand in instability in bending both e_1 and e_2 are involved and together they define a limiting condition at low values of α; it will be shown that this limit is in fact instability of the Brazier type.

7. *Euler Instability and the Associated Out-of-Phase Mode.*—For values of $\lambda\alpha d/2$ up to unity, the expansion $T \simeq \frac{1}{2}\lambda\alpha d - \frac{1}{24}(\lambda\alpha d)^3 + \frac{1}{240}(\lambda\alpha d)^5$ is accurate to about $\pm$ 5 per cent., whilst the first two terms afford this accuracy up to about $\lambda\alpha d = 1 \cdot 60$. Therefore for low values of αd, particularly if λ_1 and λ_2 are not much different from unity, the expansion $T = \frac{1}{2}\lambda\alpha d - \frac{1}{24}(\lambda\alpha d)^3$ is reasonably accurate. By substitution of this form and its reciprocal $1/T = (2/\lambda\alpha d)\{1 + \frac{1}{12}(\lambda\alpha d)^2 - \frac{1}{720}(\lambda\alpha d)^4 + \ldots\}$ in (28),

$$P = \tfrac{1}{2}\lambda_1\lambda_2(\alpha d)\{1 - \tfrac{1}{12}(\lambda_1^2 + \lambda_2^2)(\alpha d)^2 + \tfrac{1}{120}(\lambda_1^4 + \lambda_1^2\lambda_2^2 + \lambda_2^4)(\alpha d)^4 - \ldots\},$$

$$P' = \tfrac{1}{6}\lambda_1\lambda_2(\alpha d)\{1 - \tfrac{1}{60}(\lambda_1^2 + \lambda_2^2)(\alpha d)^2 + \ldots\},$$

$$Q = \tfrac{1}{24}\lambda_1^2\lambda_2^2(\alpha d)^3\{1 - \tfrac{1}{10}(\lambda_1^2 + \lambda_2^2)(\alpha d)^2 + \ldots\} - \sigma P,$$

$$Q' = (2/\alpha d)\{1 + \tfrac{1}{720}\lambda_1^2\lambda_2^2(\alpha d)^4 + \ldots\} - \sigma P',$$

$$R = \tfrac{1}{2}\lambda_1\lambda_2(\alpha d)\{1 - \tfrac{1}{120}\lambda_1^2\lambda_2^2(\alpha d)^4 + \ldots\} + 2\sigma Q + \sigma^2 P$$

and

$$R' = (2/\alpha d)\{(\lambda_1^2 + \lambda_2^2)/\lambda_1\lambda_2 + \tfrac{1}{12}\lambda_1\lambda_2(\alpha d)^2 + \ldots\} + 2\sigma Q' + \sigma^2 P';$$

also

$$\lambda_1\lambda_2 T_1 T_2 = \tfrac{1}{4}\lambda_1^2\lambda_2^2(\alpha d)^2\{1 - \tfrac{1}{12}(\lambda_1^2 + \lambda_2^2)(\alpha d)^2 + \tfrac{1}{720}(6\lambda_1^4 + 5\lambda_1^2\lambda_2^2 + 6\lambda_2^4)(\alpha d)^4 - \ldots\}$$

and

$$\lambda_1\lambda_2/T_1T_2 = (2/\alpha d)^2\{1 + \tfrac{1}{12}(\lambda_1^2 + \lambda_2^2)(\alpha d)^2 - \tfrac{1}{720}(\lambda_1^4 - 5\lambda_1\lambda_2^2 + \lambda_2^4)(\alpha d)^4 + \ldots\}.$$

Then, by substituting in formula (27)

$$e_2 \simeq \tfrac{1}{4}(\alpha d)^2\left[\frac{(s/d)^2}{3(1-\sigma_s^2)} + \frac{\tfrac{1}{6}\lambda_1\lambda_2(dE/sE_s) + 1 + (2s/d) + (s/d)^2 + \text{terms in } (\alpha d)^2 \text{ etc.}}{1 + \tfrac{1}{2}(E_s s/Ed)(\alpha d)^2\{(\lambda_1^2+\lambda_2^2)/\lambda_1\lambda_2 + 2\sigma + \text{terms in } (\alpha d)^2 \text{ etc.}\}}\right] \quad \ldots \quad \ldots \quad (36)$$

and if $(s/d)^2$ is negligible in comparison with unity,

$$e_2 \simeq \tfrac{1}{4}(\alpha d)^2\{1 + 2(s/d) + \tfrac{1}{6}(E_t d/E_s s)\}/\{1 + \tfrac{1}{2}(E_s s/G_d d)(\alpha d)^2\}\,. \quad \ldots \quad \ldots \quad \ldots \quad (36a)$$

The term $\frac{1}{6}(E_t d/E_s s)$ in the numerator of (36a) represents the resistance of the core to bending; but the tendency of the core to buckle has not been taken into account and to correct for this omission the value of e_2 given by (36a) has to be divided by $1 + \frac{1}{2}K(E_t d/E_s s)$, where K is very nearly unity (*see* R. & M. 1987[1], Appendix A). (A similar correction will be made later to be value of e_1 deduced from formula (26), but in this case the appropriate value of K is about $\frac{1}{3}$).

The corrected value of e_2 is then

$$e_2 \simeq \tfrac{1}{4}(\alpha d)^2\{1 + 2(s/d) + \tfrac{1}{6}(E_t d/E_s s)\}/\{1 + \tfrac{1}{2}(E_s s/G_d d)(\alpha d)^2\}\{1 + \tfrac{1}{2}(E_t d/E_s s)\}\,, \quad (37)$$

and this is of the form $1/P_c = 1/P_e + r$, $\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37a)$

where P_c is the critical load $= (2E_s s + E_t d)\, e_2$,
P_e is the Euler load $= \frac{1}{2}\{E_s s(1 + 2s/d) + \frac{1}{6}E_t d\}(\alpha d)^2$

and $r = 1/G_d d\{1 + 2(s/d) + \frac{1}{6}(E_t d/E_s s)\}$ is the effective shear constant.

The formula analogous to (36) derived from formula (26) is

$$e_1 \simeq \frac{(\alpha s)^2}{12(1-\sigma_s^2)} + \frac{\{(E_t d/E_s s) - \sigma\lambda_1\lambda_2(s/d)(\alpha d)^2\}\{1 - \frac{1}{12}(\lambda_1^2+\lambda_2^2)(\alpha d)^2 + \ldots\} + 2 + \frac{1}{24}(E_t s/E_d d)(2 + 3s/d)(\alpha d)^4}{(\alpha d)^2(E_s s/E_d d)\{1 - \frac{1}{2}\sigma^2(E_t d/E_s s) + \frac{1}{6}\lambda_1\lambda_2(\alpha d)^2\}}\,. \quad \ldots \quad \ldots \quad (38)$$

Assuming that (αd) is small, that $E_t d/E_s s$ is comparable with $(\alpha d)^2$ and that $(s/d)^2$ is negligible, and correcting for the tendency of the core to buckle, formula (38) leads to the approximate form

$$e_1 \simeq \frac{1 + \frac{1}{2}(E_t d/E_s s) - \{\frac{1}{6}(\lambda_1^2+\lambda_2^2)(E_t d/E_s s) + \frac{1}{2}\sigma\lambda_1\lambda_2(s/d)(\alpha d)^2}{\frac{1}{2}(\alpha d)^2(E_s s/E_d d)\{1 + \frac{1}{6}(E_t d/E_s s)\}\{1 - \frac{1}{2}\sigma^2(E_t d/E_s s)\}}\,. \quad \ldots \quad \ldots \quad (38a)$$

and for very small values of αd, assuming σ^2 to be negligible in comparison with unity

$$e_1 \simeq 2(E_d d/E_s s)/(\alpha d)^2\{1 + \tfrac{1}{6}(E_t d/E_s s)\}\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38b)$$

Then using (36a) and neglecting (s/d) in comparison with unity

$$e_1 e_2 \simeq \tfrac{1}{2}(E_d d/E_s s)/\{1 + \tfrac{1}{2}(E_s s/G_d d)(\alpha d)^2\}\{1 + \tfrac{1}{2}(E_t d/E_s s)\}\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

The factor $1 + \frac{1}{2}(E_s s/G_d d)(\alpha d)^2$ is retained because $E_s s/G_d d$ may be large, but a first approximation to the critical value of ψ when e is zero is

$$\psi_b = (E_d d/2E_s s)^{1/2}\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39a)$$

This is the condition for instability of the Brazier type.

Unfortunately any attempt to apply formula (38) to cases for which (αd) is only moderately small seems to lead to inconsistencies. These applications indicate the possibility of a true out-of-phase instability in the range of (αd) between the high value characteristic of true wrinkling and the very low value characteristic of true Euler instability; but they do not suffice to determine even a crudely approximate value for the minimal value of e_1. This possibility and particularly the question whether e_1 may ever have two minima in the whole range of α, has therefore to be discussed on the basis of the original formula (26) with the complete forms (28) for P, Q and R and without any approximation.

As an indication of a possible interpretation of formula (38) we may write

$$e_1 \simeq \frac{(\alpha s)^2}{12\,(1-\sigma_s^{\,2})} + \frac{2(E_d s/E_s d)}{(\alpha s)^2}\,. \qquad (40)$$

when the minimum value is $\{2(E_d s/E_s d)/3(1-\sigma_s^{\,2})\}^{1/2}$, ... (40a)

when $(\alpha d)^2 = 2\{6(1-\sigma_s^{\,2})\,(E_d d^3/E_s s^3)\}^{1/2}$.

This minimum is applicable only if $(\alpha d) < 2$, so that $(d/s)^3 < \frac{2}{3}\cdot(E_s/E_d)$. For example if $E_s/E_d = 12{,}000$, (s/d) must be greater than $\frac{1}{5}$ and $e_1 > \frac{1}{600}$. At such large values of αd, formula (37) reduces to $e_2 \simeq G_d d/2E_s s$, so that formula (40a) represents a practical limit only if $G_d/E_s > 1/6000$; that is G_d/E_d must be greater than 2. This condition is likely to be fulfilled only by an abnormal material.

8. *Conclusions.*—Under the combined action of a uniform compression e and a bending strain ψ, so that the compressive strains in the skins are $e \pm \psi$, a sandwich will become unstable when $\psi^2 = (e_1 - e)\,(e_2 - e)$, where e_1 and e_2 are critical strains under pure compression for out-of-phase and in-phase buckling over a common wavelength ; the ratio of ψ to e being given, the wavelength will be such as to render e or ψ a minimum.

For true wrinkling instability over a short wavelength, e_1 is only slightly greater than e_2 and distinction between them is unnecessary. In this case therefore the condition for instability is $e + \psi = e_w$, where $e_w = K(E_l E_d^{\,3}/E_s^{\,4})^{1/6}$ and E_l is the longitudinal modulus of the core material, E_d is its transverse modulus and E_s is the modulus of the skin material. The value of the constant K depends primarily on the ratio of the thickness s of the skins to the thickness d of the core but also to some extent on the ratios between the elastic constants of the core. If $s/d < \frac{1}{100}$, the value of the constant K varies very little from $0{\cdot}6$.

In cases of instability over longer wavelengths e_1 is likely to be appreciably greater than e_2. In that case, assuming $\psi/e = k$ (const), the condition for instability may be written

$$\frac{1}{e^2} - \left(\frac{e_1 + e_2}{e_1 e_2}\right)\frac{1}{e} + \frac{1 - k^2}{e_1 e_2} = 0\,;$$

that is $$\frac{1}{e} = \frac{1}{2e_1 e_2}\{e_1 + e_2 \pm \sqrt{[(e_1 - e_2)^2 + 4k^2 e_1 e_2]}\}\,,$$

or $$\frac{1}{e} \simeq \frac{1}{2e_1 e_2}\{e_1 + e_2 \pm (e_1 - e_2 + 2k^2 e_2)\} \text{ assuming } e_1 >> e_2 \text{ and } k < 1 \text{ say.}$$

The least value of e is then given by $1/e = 1/e_2 + k^2/e_1$, so that $e < e_2$; but the strain in the more heavily loaded skin $= (1 + k)e = (1 + k)\,e_1 e_2/(e_1 + k^2 e_2)$ and this is greater than e_2.

Under pure bending, instability occurs when $\psi = (e_1 e_2)^{1/2}$ and over long wavelengths this condition represents a kind of instability of the Brazier type.

9. *Further Developments.*—Some further analysis is required of the types of instability characterized by moderate wavelengths, which may occur in sandwiches with relatively thick skins ($d/s < 100$). In this analysis it may probably be necessary to examine a wide range of elastic properties of the core material.

Apart from the uncertainty in this field intermediate between true wrinkling and true Euler instability, present information as to the conditions for instability appear adequate ; but standard means of measurement of the elastic properties of the core and particularly of its shear modulus need to be adopted. Then by tests on a carefully chosen series of sandwich struts the values of the constants in the theoretical formulae should be checked.

In the theoretical field there is need to extend the theory of sandwich panels both flat and curved to aeolotropic core materials.

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APPENDIX

Application of Theory of Part III to Steel-Formvar Sandwich Struts by J. R. Riddell, B.Eng.

In order to provide an illustration of the application of the paper, a fairly extensive series of computations has been made for one particular range of sandwich struts. These struts consist of expanded formvar core and steel skins, with the following properties :—

Young's modulus of skin material	$E_s = 30 \times 10^6$ lb./sq. in.
Young's modulus of core material longitudinally	$E_l = 22{,}000$ lb/sq. in.
Young's modulus of core material transversely	$E_d = 1{,}800$ lb./sq. in.
Shear Modulus of core material	$G_d = 2{,}800$ lb./sq. in.
Thickness of core	$d = 0 \cdot 492$ in.

The Poisson's ratio of the core material has been taken to be $\sigma = 0 \cdot 124$ and four values of the skin thickness s, namely 0·00100, 0·00288, 0·01016 and 0·02000 in. have been considered.

This range of struts includes the set described in R. & M. 2143[4], and the moduli tabled above are taken from this paper, as are also the value of d and the values of 0·00288 and 0·01016 in. of s ; the values of 0·001 and 0·020 in. of s are included here in order to provide a more complete illustration of the effect of skin thickness. The value of Poisson's ratio for the expanded formvar was not accurately known ; the value of 0·124 has been used because this value gives $\lambda_1 = \lambda_2$ so that the computations were slightly simplified. As a check some cases were computed with $\sigma = 0$, but this change made no material difference to the results.

Fig. 2 shows the values of e_1 and e_2 plotted against αs for the four values of skin thickness. It is apparent that for this range of sandwich struts at least, e_1 has but one minimum over the complete range of αs, and that e_1 is always greater than e_2. At high values of αs, however, when both T_1 and T_2 are indistinguishable from unity, e_1 and e_2 are almost exactly equal. When d/s is greater than about 40, e_2 at first increases with αs, reaches a maximum, falls to a minimum, and finally increases slowly. The minimum value of e_2 (the wrinkling strain) is very nearly constant at about 0·00150 and occurs when αs is about 0·072; these figures agree well with the approximate values $e_2 \simeq 0 \cdot 00145$ and $\alpha s \simeq 0 \cdot 073$ obtained from equations (33c) and (34d). When d/s is less than about 40 both the maximum and minimum values of e_2 are suppressed, and the value of e_2 increases continuously with αs ; moreover e_1 and e_2 do not become approximately equal until αs is appreciably greater than the value indicated by equation (34d).

The part of the e_2 curve at low values of αs represents buckling in-phase over a long wavelength ; that is, modified Euler instability. Therefore the incidence of this type of instability is most conveniently represented in terms of the half-wavelength $l = \pi/\alpha$, which may be regarded as the length of the (pin-ended) strut. If the strut is short, the value of $\alpha = \pi/l$ may correspond to a value of e_2 greater than its minimum value e_w for wrinkling ; but for long struts the Euler mode may give the lower value of e_2. Thus in Fig. 3, as the strut length l is decreased, the critical strain for struts with $d/s > 40$ increases until a strain equal to the wrinkling strain e_w is reached : then the critical strain remains constant at e_w until αs ($= \pi s/l$) exceeds its value at the minimum (wrinkling) strain in Fig. 2. For still shorter struts the stress in the skins would increase until the limiting compressive strength of the skin material might be reached ; but even with thick skins (0·01 in.) this increase above the wrinkling stress affects only struts shorter than $\frac{1}{2}$ in., so that the effect has no practical significance. When $d/s < 40$ there is no true wrinkling condition and the critical increases continuously as l is decreased. At the same time, the tendency to wrinkle is indicated by a partial " arrest " in the curve.

The approximate expression for e_2 with small values of α (long wavelengths) given in equation (37) has been computed for the four values of d/s examined. These approximate values, when e_2 is less than e_w, are always less than the values obtained from the exact expression, equation (27), but the percentage differences for values of d/s of 492, 170·8 and 48·4 are not considerable, being about 15, 7 and 3 per cent. respectively. When e_w is less than e_2, wrinkling type failure will occur and equation (37) ceases to have any significance. When $s = 0 \cdot 020$ in. ($d/s = 24 \cdot 6$) the percentage error is very small (less than 3 per cent.) until αs equals about $0 \cdot 03$; then, after the " arrest " indicated in Fig. 3, the exact value of e_2 begins to increase appreciably, while the approximate e_2 tends to a limiting value of $0 \cdot 00124$.

It is therefore apparent that, for values of d/s greater than about 40, a sufficiently accurate approximation to the critical strain in both Euler and wrinkling modes may be obtained from equations (37) and (33c) respectively.

In Fig. 4, p, the load per unit width carried by the strut, is plotted against the length of strut l on a log-log basis. For simplicity it has been assumed that the skins carry all the direct compressive load on the strut. Then $p = 2sf = 2sE_s e$, where f is the compressive stress in the skins ; correction for the load carried by the core would slightly raise the curves, particularly those for the thinner skins. In Fig. 5 this stress f is shown plotted against the structure loading coefficient p/l ; the wide range of values of p/l from $0 \cdot 01$ lb./in./in. to 10,000 lb./in/.in. has been covered in order to illustrate the variations in f over a comprehensive range. The majority of practical design cases are likely to fall in the range 1 to 100 lb./in. in.

In Fig. 6 the results otherwise shown in Fig. 3 have been replotted for struts of various lengths on the ordinates $(s/d)\ (E_s/E_d)^{1/3}$ and $f/(E_d^2 E_s)^{1/3}$ used by Gough, Elam, de Bruyne, 1939[3]. Since computed values were available for only four values of the skin thickness, the curves so plotted, particularly for the shorter struts, may be appreciably in error. It is apparent, however, that the curves are generally similar in form to those plotted in Ref. 3, Figs. 13 and 14. To obtain a closer check on the shape of the curves in Fig. 6 many more values of the skin thickness would have to be considered. It is worth remark in both Figs. 3 and 6 that Euler instability of long struts is independent of the ratio s/d.

It must be emphasised that Figs. 2 to 6 deal with only one special set of sandwich struts with known materials, dimensions and moduli ; these curves afford one illustration but not a complete picture of the results of the paper. It is probable that the results are typical, but until this probability has been investigated over a wider range of cases, the possibility that different materials may exhibit radically different properties must be borne in mind.

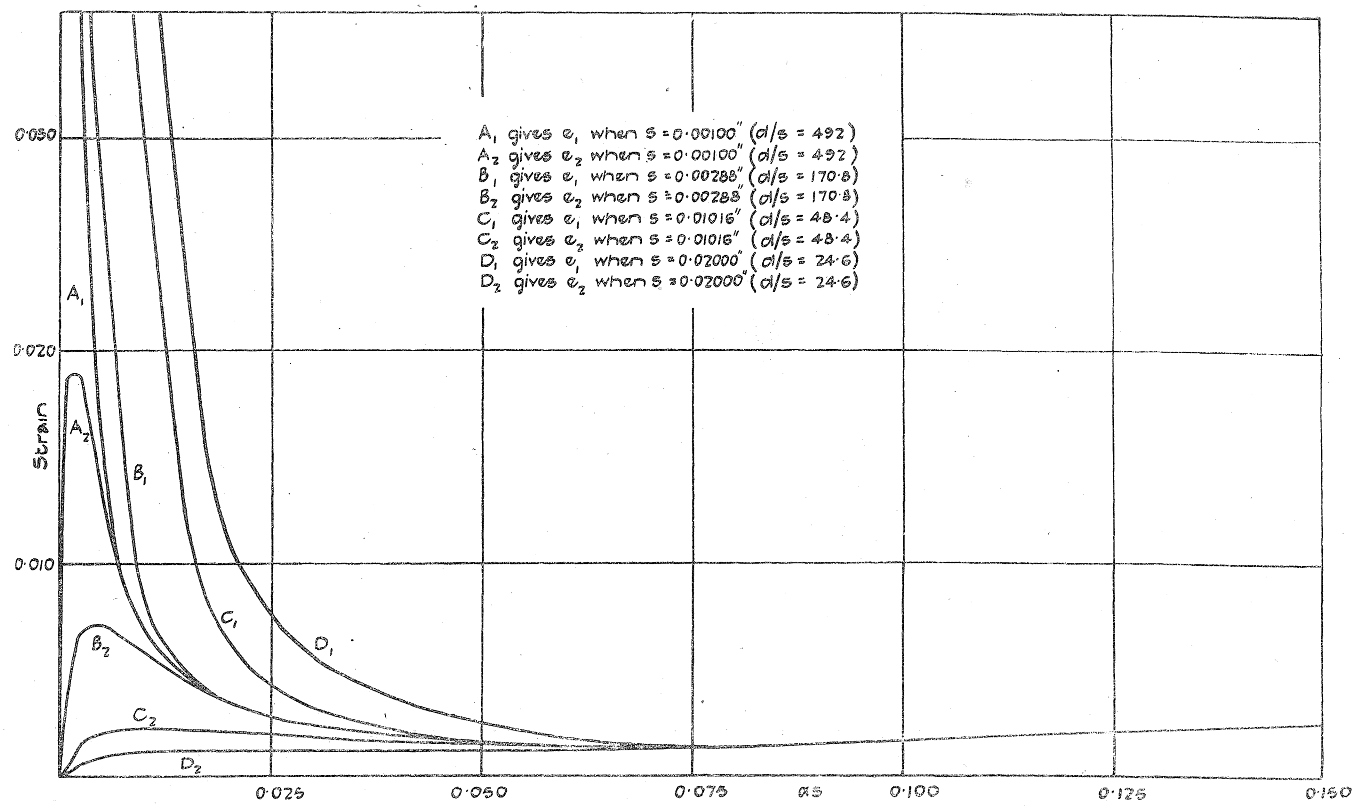

FIG. 2. e_1 and e_2 Plotted against αs for Varying Skin Thickness.

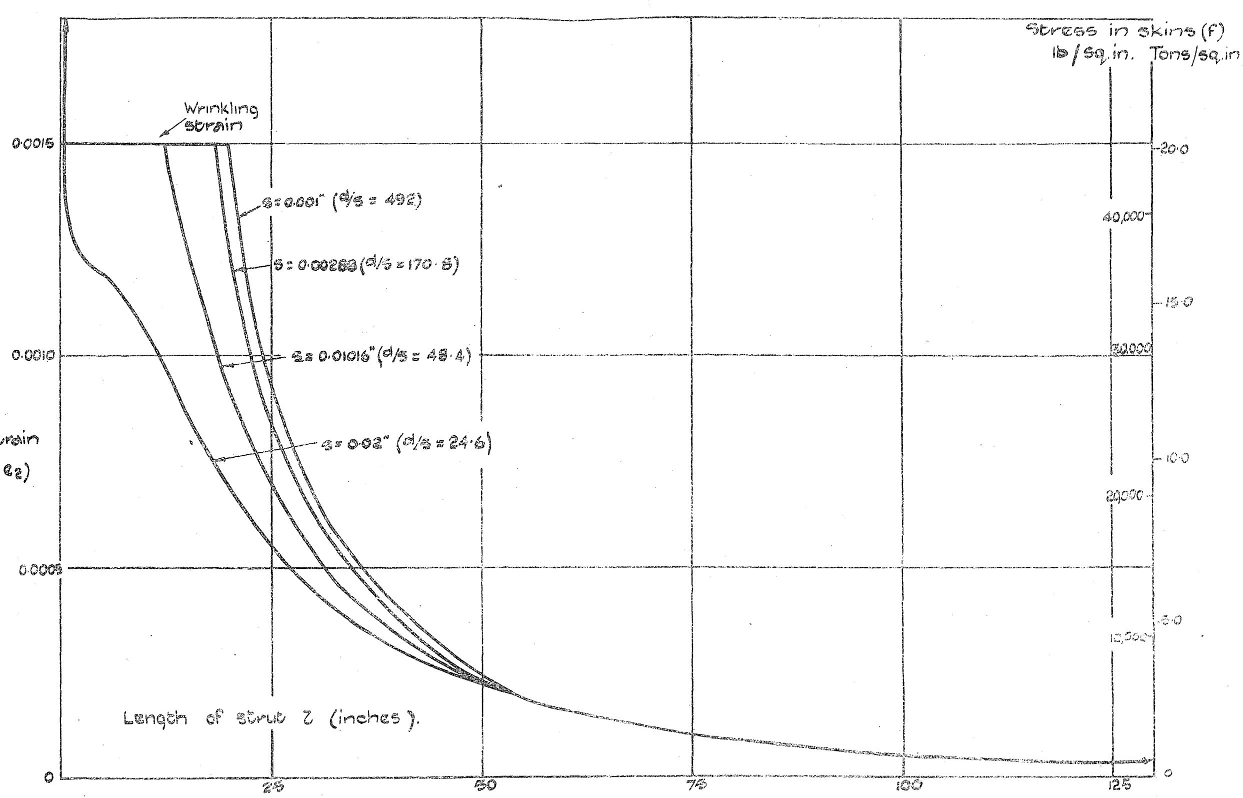

FIG. 3. e_2 and f Plotted against l for Varying Skin Thickness.

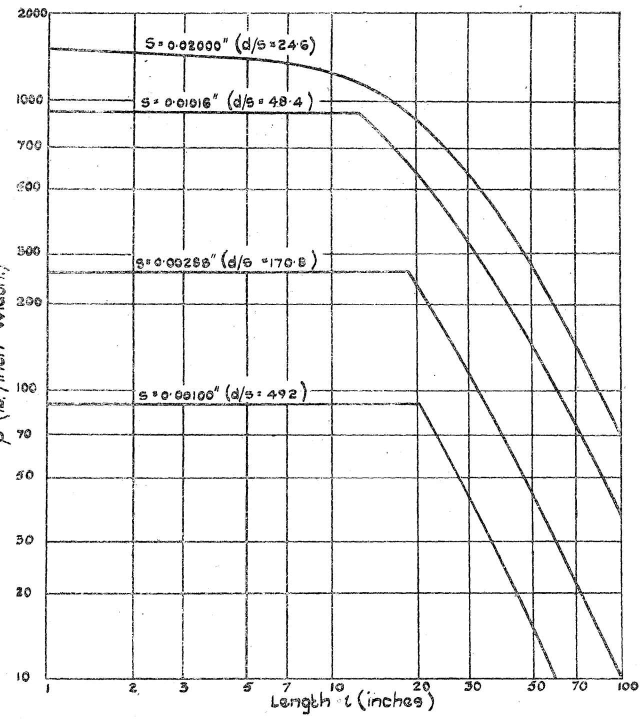

FIG. 4. p Plotted against l for Varying Skin Thickness.

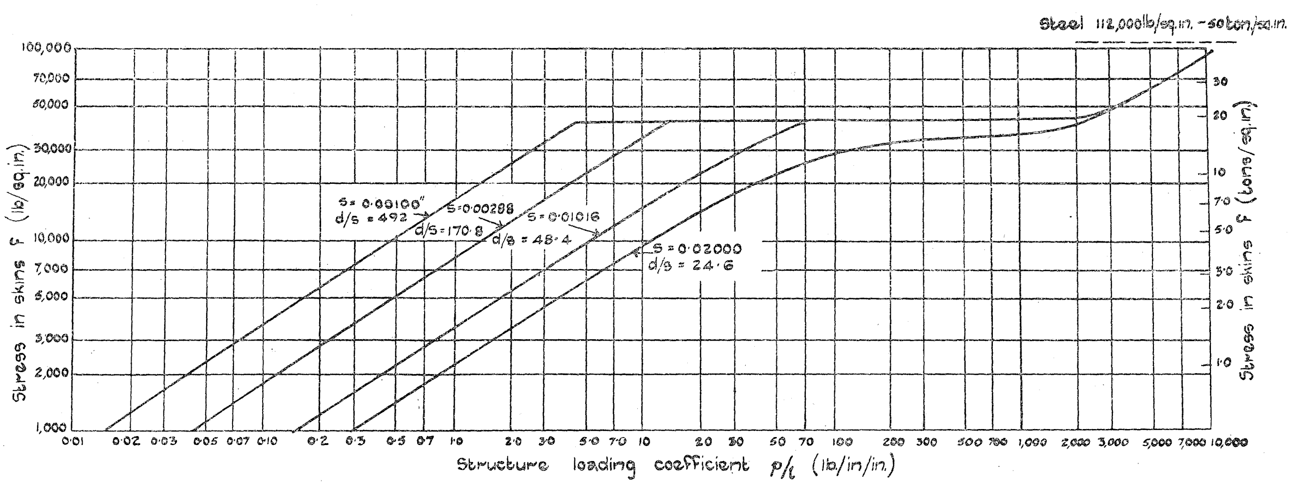

FIG. 5. f Plotted against p/l for Varying Skin Thickness.

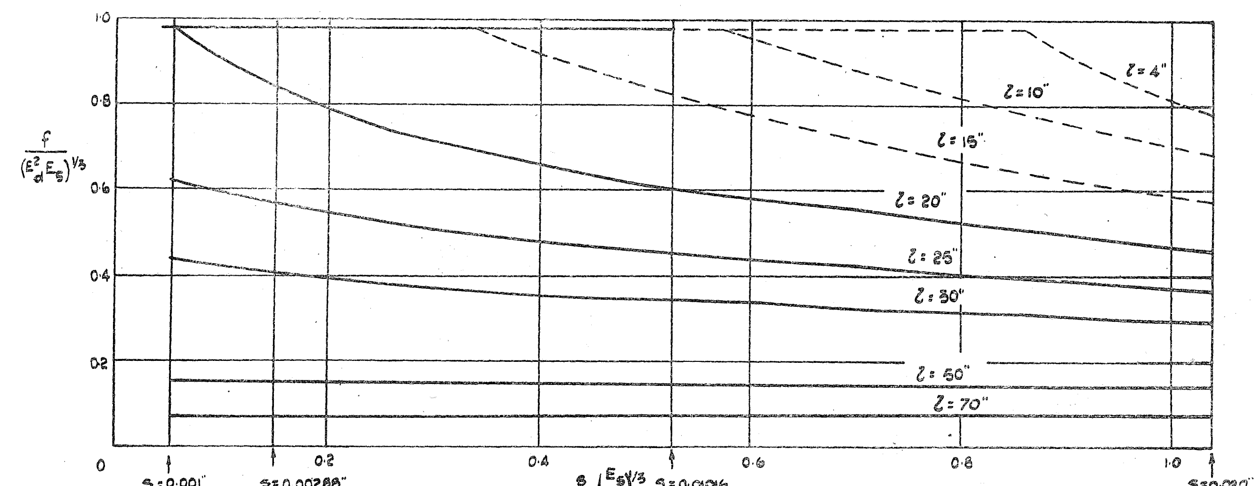

FIG. 6. $f/(E_d^2 E_s)^{1/3}$ Plotted against $(s/d)\ (E_s/E_d)^{1/3}$ for Varying Lengths of Strut.

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