C.P. No. 242
(17,569)
A.R.C. Technical Report

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MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
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Loading Conditions following an Automatic Pilot Failure (Rudder Channel)

By

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LONDON: HER MAJESTY'S STATIONERY OFFICE

1956

FIVE SHILLINGS NET

C.P. No. 242

U.D.C. No. 629.13.014.59 : 004.63 : 533.6.013.8

Technical Note No. Structures 154

February, 1955

ROYAL AIRCRAFT ESTABLISHMENT

Loading Conditions Following an Automatic Pilot Failure (Rudder Channel)

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SUMMARY

A method is presented for the determination of the critical loading conditions of aircraft that ensue from an automatic pilot failure in the rudder channel.

General expressions have been derived through response theory for the angle of sideslip, fin-and-rudder load and lateral acceleration both at the C.G. of the aircraft and at the tail, that result from the sequence of rudder movements assumed to follow an automatic pilot failure. Analysis of these general expressions leads to formulae suitable for assessing the numerical values of the critical loading conditions and it is suggested that these formulae might form a basis for the interpretation of the appropriate design requirement[1].

An example is given to illustrate the type of response produced by a rudder channel failure and the calculation procedure.

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1 Introduction

A procedure has recently been developed[2] for assessing the critical loading conditions that ensue from a failure in the elevator channel of an automatic pilot. The aim of the present investigation is to develop an analogous procedure for the rudder channel failure. Owing to the numerical differences in the magnitude of the various parameters, numerous minor differences in approach have, however been necessary. A full discussion of the problem is given below.

The critical conditions in the present case are usually associated with the maximum angle of sideslip and with the maximum aerodynamic load on the fin-and-rudder.

2 Details of the investigation

The investigation has been split into three sections

(i) the choice of a general rudder time-history to describe the sequence of movements that occur after a failure

(ii) the derivation of the response of the aircraft to the chosen rudder movements

(iii) the analysis of the response to determine the numerical values of the parameters of the rudder movement which produce the critical loading conditions and to obtain workable formulae and charts suitable for routine estimation of these conditions.

These sections are described and discussed in the following paragraphs.

2.1 The rudder movement

The general sequence of events in a rudder channel failure is the same as that in an elevator channel failure[2]. Accordingly, it is assumed that the rudder time-history is composed of the following stages

(i) "runaway", during which the rudder moves at a constant rate corresponding to the maximum rate of the servomotor.

(ii) "check" after the rudder has been arrested at its maximum displacement, either by a stop or by the fact that aerodynamic forces on the rudder have stalled the servomotor, until

(iii) "recovery" when the rudder has been moved back instantaneously.

The quantities defining this sequence of rudder movement are - the "runaway" rate, the maximum displacement during the "check" period, the time of initiation of the recovery action and the displacement in the recovery. The sequence is illustrated diagramatically in Fig.1. The timing of the recovery action and the magnitude of the maximum displacement in the "check" period and in the "recovery" are discussed in Appendix I paragraphs A5, A7 and A8 respectively.

The only difference between the sequence defined above, and that used in the investigation of the elevator channel failure[2], is the adoption of instantaneous movement for the recovery action instead of gradual movement. This change permits a simplification in the mathematical treatment without affecting the accuracy of the final results to any appreciable extent; the simplification leads to a slightly conservative result.

2.2 The response of the aircraft

Manual operation of the rudder alone causes the aircraft to sideslip and roll, and perhaps pitch. However, in the event of a failure in the rudder channel of the automatic pilot, it may be assumed that the aileron and elevator channels continue to function correctly until the automatic pilot is disengaged, and operate so as to reduce the rolling and pitching motions induced by the rudder displacement. As a result, the aircraft tends to execute a flat turn.* The flat turn involves sideslip and lateral acceleration, and both these movements affect the loading on the fin-and-rudder. The loading is due to both the disturbing movement of the rudder and the ensuing response of the aircraft itself.

The response of an aircraft to the sequence of rudder movements assumed to follow the failure is derived in Appendix I by means of Laplace Transforms. Expressions are obtained for the incremental values of the angle of sideslip (para. A2), fin-and-rudder load (para. A3) and lateral acceleration both at the C.G. of the aircraft and at the tail (para. A4). The response in these quantities is illustrated in Figs. 2, 3 and 4 respectively; the data for these examples are contained in Table I.

2.3 The critical conditions

The various response formulae of Appendix I paragraphs A2, A3 and A4 may be used to determine the complete time-histories of the loads and accelerations produced by the sequence of rudder movements defined in para. 2.1. However, from the airworthiness aspect it is the various local maxima of these quantities, and in particular their absolute maxima which are of major interest. It is therefore necessary to analyse the response formulae and obtain general expressions for the magnitudes of the local maxima. Details of such an analysis are given in Appendix I para. A5, including some discussion as to the exact timing of the recovery action, which is not precisely defined in para. 2.1 and which must be chosen in such a way that the absolute maxima of the various quantities are realised; use is made of Figs. 2, 3 and 4 to illustrate this discussion. Information concerning a number of charts which may be used to facilitate the calculation of the absolute maxima is given in Appendix I para. A6. With regard to these charts, it has been found that the number needed may be reduced to three if certain approximations are made. These approximations, have no significant effect on the final results.

The analysis of the response formulae indicates that the sequence of rudder movements defined in para. 2.1 produces two significant local maxima of each of the response quantities mentioned above (cf Figs. 2, 3 and 4) and thus, provided the sequence of rudder movements is chosen realistically, these maxima represent the design conditions for the case of an automatic pilot failure in the rudder channel.

In Appendix II, the procedure for calculating the absolute maxima is presented in such a form that the calculations can be carried out directly by a computer.

3 Concluding remarks

This note presents a rational method for assessing the loads on aircraft following an automatic pilot failure in the rudder channel. It is suggested that it might form a basis for the interpretation of the relevant design requirement[1].

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* At least, if the automatic pilot heading control is coupled to the rudder.

NOTATION

A, B, C, C_1	coefficients, see equation (23)
$a_1 = -\frac{\partial C_{Yf}}{\partial \beta}$	including effects of local sidewash at the tail
$a_2 = \frac{\partial C_{Yf}}{\partial \zeta}$	
b	wing span
$b_1 = -\frac{\partial C_h}{\partial \beta}$	including effects of local sidewash at the tail
$b_2 = \frac{\partial C_h}{\partial \zeta}$	
C_h	rudder hinge moment coefficient
C_{h_s}	rudder hinge moment coefficient equivalent to the stalling torque of the rudder servomotor
C_{Yf}	lateral force coefficient of the fin-and-rudder
$\left(\frac{d\zeta}{dt}\right)_f$	"runaway" rate of the rudder
E	coefficient, see equation (25)
G	special function of time, see equation (11)
g	acceleration due to gravity
$H, \overline{H}_f, \overline{H}_c$	special functions of time, see equations (13), (29) and (40) respectively
$i_c = \frac{4k_c^2}{b^2}$	coefficient of inertia about the z axis
J	non-dimensional circular frequency, see equation (4)
$J\tau_a, J\tau_b, J\tau_a'$ etc.	non-dimensional times of occurrence (in radians) of local maxima, see Appendix I paragraphs A5 and A6
$J\tau_f$	non-dimensional time (in radians) at which the "runaway" is arrested
$J\tau_c$	non-dimensional time (in radians) at which the "recovery" is initiated
K	special function of time, see equation (12)
K_a	coefficient, see equation (5)
K_π, K_π'	see equation (33) and (41) respectively

NOTATION (Contd)

k	see equation (18)
k_c	radius of gyration about the z axis
$L, \bar{L}_f, \bar{L}_c$	special function of time, see equations (14), (29) and (40) respectively
ℓ	fin-and-rudder arm
ℓ_R	distance of C.P. of fin-and-rudder load due to rudder deflection to C.G. of the aircraft
n_s	coefficient of lateral acceleration at the C.G. of the aircraft
n_ℓ	coefficient of lateral acceleration at the tail of the aircraft due to acceleration in yaw
n_t	coefficient of total lateral acceleration at the tail
n_r	rotary damping derivative
n_v	static stability derivative in sideslip
P	net fin-and-rudder load
r	angular velocity in yaw
$\hat{r} = r\hat{t}$	non-dimensional angular velocity in yaw
S	wing area
S''	fin-and-rudder area
t	time in seconds
$\hat{t} = \frac{W}{g\rho SV}$	unit of aerodynamic time
V	velocity of the C.G. of the aircraft
$\bar{V}_R = \frac{S''\ell_R}{Sb}$	fin-and-rudder volume coefficient
W	weight of aircraft
y_v	lateral force derivative due to changes in angle of sideslip
$\bar{y}_v = - y_v$	
y_ζ	lateral force derivative due to changes in rudder angle
Z, Z_1, Z_2	see equations (36) and (40)
β	angle of sideslip

NOTATION (Contd)

$\delta_n = \frac{1}{i_c} \cdot \mu_2 \bar{V}_R a_2$	rudder effectiveness
ζ	rudder displacement
ζ_c	rudder displacement during "recovery"
ζ_f	rudder displacement during "runaway"
$\bar{\zeta}$	limit of rudder displacement under automatic pilot action
$\Lambda_a, \Lambda_b, \Lambda_o$	functions, see equations (48) and (51)
$\mu_2 = \frac{2W}{g\rho Sb} = \frac{2\ell}{b} \cdot \mu_3$	relative density of aircraft (referred to semi span)
$\mu_3 = \frac{W}{g\rho S\ell} = \frac{b}{2\ell} \cdot \mu_2$	relative density of aircraft (referred to fin-and-rudder arm)
$\nu_n = -\frac{1}{i_c} \cdot n_r$	rotary damping parameter
Π_a, Π_b	functions, see equation (33)
Π'_a, Π'_b	functions, see equation (41)
ρ	air density
φ	see equation (19)
$\tau = \frac{t}{\hat{t}}$	non-dimensional aerodynamic time
$\omega_n = \frac{1}{i_c} \cdot \mu_2 \cdot n_v$	static stability parameter in sideslip

Suffices

a	relating to the first maxima (in time) of the critical conditions
b	relating to the second maxima (in time) of the critical conditions
f	associated with point at which the "runaway" is arrested
c	associated with the "recovery"
β	due to the effective incidence of the fin-and-rudder
ζ	due to rudder displacement

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APPENDIX I

The mathematical analysis

A1 Equations of motion

The non-dimensional linearised differential equations of lateral motion of an aircraft when rolling and pitching motions are absent may be written (cf Ref.3)

$$\frac{d\beta}{d\tau} + \bar{y}_v\beta + \hat{r} = 0 \tag{1a}$$

$$-\omega_n\beta + \frac{d\hat{r}}{d\tau} + \nu_n\hat{r} = -\delta_n\zeta \tag{1b}$$

These equations may be written in terms of β alone, whence

$$\frac{d^2\beta}{d\tau^2} + (\nu_n + \bar{y}_v)\frac{d\beta}{d\tau} + (\omega_n + \bar{y}_v\nu_n)\beta = \delta_n\zeta \tag{2}$$

or

$$\frac{d^2\beta}{d\tau^2} + 2R\frac{d\beta}{d\tau} + (R^2 + J^2)\beta = \delta_n\zeta \tag{3}$$

where $R = \frac{1}{2}(\nu_n + \bar{y}_v)$ is the non-dimensional damping factor of the lateral oscillations of the aircraft and $J = \sqrt{\omega_n - \frac{1}{4}(\nu_n - \bar{y}_v)^2}$ is the non-dimensional frequency factor of the lateral oscillations of the aircraft. (4)

A2 Solution of the angle of sideslip

The rudder time history defined in para. 2.1 is composed of linear and instantaneous movements, and the solution to equation (3) is therefore required for both these types of movement; the principle of superposition may then be used to build up the solution for the complete sequence of rudder movements (cf Ref.2).

For linear movement of the rudder the solution is, using the notation of Fig.1

$$\beta = \frac{\delta_n}{J^2}\left(\frac{\zeta_f}{J\tau_f}\right) \cdot K_a^2 \left(\frac{J\tau}{K_a} + 2\frac{R}{J}\left(e^{-\frac{R}{J}J\tau}\cos J\tau - 1\right) + \left(\frac{R^2}{J^2} - 1\right)e^{-\frac{R}{J}J\tau}\sin J\tau\right) \tag{5}$$

where $K_a = \dfrac{1}{\left(\dfrac{R}{J}\right)^2 + 1}$

Whence

$$\frac{d\beta}{d\tau} = \frac{\delta_n}{J^2} \cdot \left(\frac{\zeta_f}{J\tau_f}\right) \cdot J \cdot K_a \left(1 - e^{-\frac{R}{J}J\tau} \cos J\tau - \frac{R}{J} e^{-\frac{R}{J}J\tau} \sin J\tau\right) \quad (6)$$

and

$$\frac{d^2\beta}{d\tau^2} = \frac{\delta_n}{J^2}\left(\frac{\zeta_f}{J\tau_f}\right) \cdot J^2 \cdot e^{-\frac{R}{J}J\tau} \sin J\tau \quad (7)$$

and for instantaneous movement of the rudder the solution is, with the notation of Fig.1

$$\beta = \frac{\delta_n}{J^2} \cdot \zeta_c \cdot K_a \left(1 - e^{-\frac{R}{J}J\tau} \cos J\tau - \frac{R}{J} e^{-\frac{R}{J}J\tau} \sin J\tau\right) \quad (8)$$

so that

$$\frac{d\beta}{d\tau} = \frac{\delta_n}{J^2} \cdot \zeta_c \cdot J \cdot e^{-\frac{R}{J}J\tau} \sin J\tau \quad (9)$$

and

$$\frac{d^2\beta}{d\tau^2} = \frac{\delta_n}{J^2} \cdot \zeta_c \cdot J^2 \left(e^{-\frac{R}{J}J\tau} \cos J\tau - \frac{R}{J} e^{-\frac{R}{J}J\tau} \sin J\tau\right) \quad (10)$$

Thus, using the following special functions of time

$$G = K_a^2 \left(\frac{J\tau}{K_a} + 2\,\frac{R}{J}\left(e^{-\frac{R}{J}J\tau} \cos J\tau - 1\right) + \left(\left(\frac{R}{J}\right)^2 - 1\right) e^{-\frac{R}{J}J\tau} \sin J\tau\right) \quad (11)$$

$$K = K_a \left(1 - e^{-\frac{R}{J}J\tau} \cos J\tau - \frac{R}{J} e^{-\frac{R}{J}J\tau} \sin J\tau\right) \quad (12)$$

$$H = e^{-\frac{R}{J}J\tau} \cos J\tau \quad (13)$$

$$L = e^{-\frac{R}{J}J\tau} \sin J\tau \quad (14)$$

the complete solution for β in Stage III, i.e. when the recovery action has been taken, is

$$\beta = \frac{\delta_n}{J^2} \cdot \left(\frac{\zeta_f}{J\tau_f}\right)\left(G - G_f - k\,K_c\right) \qquad (15)$$

whence

$$\frac{d\beta}{d\tau} = \frac{\delta_n}{J^2} \cdot \left(\frac{\zeta_f}{J\tau_f}\right) \cdot J \cdot (K - K_f - k\,L_c) \qquad (16)$$

and

$$\frac{d^2\beta}{d\tau^2} = \frac{\delta_n}{J^2} \cdot \left(\frac{\zeta_f}{J\tau_f}\right) J^2 \left(L - L_f - k\left(H_c - \frac{R}{J} L_c\right)\right) \qquad (17)$$

where $$k = \varphi J \tau_f \qquad (18)$$

and $$\varphi = -\zeta_c/\zeta_f \qquad (19)$$

The suffices to the special functions, f and c, denote that $J\tau$ is everywhere replaced by $J(\tau - \tau_f)$ and $J(\tau - \tau_c)$ respectively in these suffixed functions. The corresponding complete solutions in Stages II and I may be obtained from equation (15) etc. by putting equal to zero all the special functions with suffix c and f and c respectively. Thus in Stage I the appropriate equation for β is

$$\beta = \frac{\delta_n}{J^2}\left(\frac{\zeta_f}{J\tau_f}\right) G$$

and in Stage II is

$$\beta = \frac{\delta_n}{J^2}\left(\frac{\zeta_f}{J\tau_f}\right)(G - G_s)$$

The appearance of the suffixed special functions implies changes in the form of the rudder movement.

A3 Net aerodynamic load on the fin-and-rudder (cf Ref.3)

$$P = A\left(-B\beta - C\frac{d\beta}{d\tau} + a_2\zeta\right) \qquad (20)$$

$$= -AB\left(\beta + C_1 \cdot \frac{1}{J} \cdot \frac{d\beta}{d\tau}\right) + Aa_2\zeta \qquad (21)$$

where

$$A = \tfrac{1}{2}\rho V^2 S'' \qquad B = \left(1 + \frac{\bar{y}_v}{\mu_3}\right) a_1$$

$$C = \frac{1}{\mu_3} a_1 \qquad C_1 = \frac{CJ}{B} \qquad (22)$$

and

$$\mu_3 = \frac{W}{g\rho S\ell}$$

It is convenient to write equation (20) in the form

$$P = P_\beta + P_\zeta$$

where

$$P_\beta = - AB\left(\beta + \frac{C_1}{J} \cdot \frac{d\beta}{d\tau}\right) \qquad (23)$$

= aerodynamic load due to effective angle of incidence of the fin

and

$$P_\zeta = Aa_2\zeta \qquad (24)$$

= aerodynamic load due to the rudder displacement.

A4 Lateral accelerations (cf Ref.3)

$$n_s \simeq - E\,(\bar{y}_v\beta - y_\zeta\zeta) \qquad (25)$$

where

$$E = 2 \cdot \frac{\tfrac{1}{2}\rho V^2}{W/S}$$

and

$$n_\ell \simeq \frac{E}{\mu_3}\left(\frac{d^2\beta}{d\tau^2} + \bar{y}_v \frac{d\beta}{d\tau} - y_\zeta \frac{d\zeta}{d\tau}\right) \qquad (26)$$

so that

$$n_t = n_s + n_\ell \qquad (27)$$

A5 Critical conditions

Examples of the response in angle of sideslip, fin-and-rudder load and lateral acceleration both at the C.G. of the aircraft and at the tail to the type of rudder movement defined in para. 2.1 are shown in Figs. 2, 3 and 4 respectively. In each case the full curve illustrates the effects of the sequence of runaway and check, and the additional curves give the effect of the recovery action. The dotted curves relate to the case in which the recovery action is timed to produce the critical conditions in the particular quantity. The data for these graphs are contained in Table I.

A5.1 Angle of sideslip (see Fig. 2)

Two local maxima occur during the sequence of runaway and recovery, and the critical values of these maxima, β_a and β_b, result when the recovery is initiated immediately the angle of sideslip reaches a mathematical peak in Stage II, i.e. when $d\beta/d\tau = 0$ in this stage. Thus $J\tau_c = J\tau_a$, where $J\tau_a$ is the first root beyond $J\tau_f$ of the equation

$$J\tau = \tan^{-1}\left\{\frac{\left(\frac{R}{J}\right)\overline{L}_f - \left(\overline{H}_f - 1\right)}{\overline{L}_f + \left(\frac{R}{J}\right)\left(\overline{H}_f - 1\right)}\right\} \tag{28}$$

where

$$\left.\begin{aligned} \overline{L}_f &= e^{\frac{R}{J}J\tau_f}\sin J\tau_f \\ \text{and}\quad \overline{H}_f &= e^{\frac{R}{J}J\tau_f}\cos J\tau_f \end{aligned}\right\} \tag{29}$$

Then (cf equation (15))

$$\beta_a = \frac{\delta_n}{J^2}\left(\frac{\zeta_f}{J\tau_f}\right)\left(G - G_f\right)_{J\tau=J\tau_a} \tag{30}$$

and

$$\beta_b = \frac{\delta_n}{J^2}\left(\frac{\zeta_f}{J\tau_f}\right)\left(G - G_f - k\,K_c\right)_{J\tau=J\tau_b} \tag{31}$$

where

$$J\tau_b = \pi + J\tau_a \tag{32}$$

However introducing the functions

$$\left.\begin{aligned} \Pi_a &= \frac{1}{J\tau_f}(G - G_f)_{J\tau=J\tau_a} \\ \Pi_b &= \frac{1}{J\tau_f}(G - G_f)_{J\tau=\pi+J\tau_a} \\ K_\pi &= K_{J\tau=\pi} \end{aligned}\right\} \qquad (33)$$

equations (30) and (31) may be simplified to

$$\beta_a = \frac{\delta_n}{J^2} \cdot \zeta_f \cdot \Pi_a \qquad (34)$$

$$\beta_b = \frac{\delta_n}{J^2} \zeta_f \cdot (\Pi_b - \varphi K_\pi) \qquad (35)$$

A5.2 Fin-and-rudder load (see Fig.3)

Here three local maxima occur during the sequence of runaway and recovery, but the first one is normally very small in magnitude and therefore unimportant from the strength aspect. The critical values of the remaining maxima, P_a and P_b, result when the recovery is initiated immediately the fin-and-rudder load reaches a maximum in Stage II, i.e. when $dP/d\tau = 0$ in this stage. Thus, in this case $J\tau_c = J\tau_a'$, where $J\tau_a'$ is the first root beyond $J\tau_f$ of the equation

$$J\tau = \tan^{-1}\left\{\frac{Z\overline{L}_f - (\overline{H}_f - 1)}{\overline{L}_f + Z\,(\overline{H}_f - 1)}\right\} \qquad (36)$$

where

$$Z = \frac{R}{J} - \frac{C_1}{K_a}$$

Then (cf equations (23) and (24))

$$P_a = P_{\beta a} + P_{\zeta a}$$

$$= -AB\frac{\delta_n}{J^2}\left(\frac{\zeta_f}{J\tau_f}\right)\Big((G - G_f) + C_1\,(K - K_f)\Big)_{J\tau=J\tau_a'} + A\,a_2\,\zeta_f\,(1 - \varphi) \qquad (37)$$

and

$$P_b = P_{\beta b} + P_{\zeta b}$$

$$= -AB\,\frac{\delta_n}{J^2}\left(\frac{\zeta_f}{J\tau_f}\right)\left((G-G_f-k\,K_c)+C_1(K-K_f-k\,L_c)\right)_{J\tau=J\tau_b'} + Aa_2\zeta_f\,(1-\varphi) \tag{38}$$

where $J\tau_b'$ is the first positive root beyond $J\tau_c$ of the equation (obtained from the condition $dP/d\tau = 0$ in Stage III)

$$J\tau = \tan^{-1}\left\{\frac{Z\overline{L}_f - (\overline{H}_f - 1) + Z_1\,\overline{L}_c + Z_2\,\overline{H}_c}{\overline{L}_f + Z(\overline{H}_f - 1) - Z_2\,\overline{L}_c + Z_1\,\overline{H}_c}\right\} \tag{39}$$

where

$$\left.\begin{aligned} Z_1 &= \frac{k}{K_a}\left(C_1\left(\frac{R}{J}\right) - 1\right) & \overline{H}_c &= e^{\frac{R}{J}J\tau_c}\cos J\tau_c \\ Z_2 &= \frac{kC_1}{K_a} & \overline{L}_c &= e^{\frac{R}{J}J\tau_c}\sin J\tau_c \end{aligned}\right\} \tag{40}$$

Introducing the following factors

$$\left.\begin{aligned} \Pi_a' &= \frac{1}{J\tau_f}\left((G - G_f) + C_1\,(K - K_f)\right)_{J\tau=J\tau_a'} \\ \Pi_b' &= \frac{1}{J\tau_f}\left((G - G_f) + C_1\,(K - K_f)\right)_{J\tau=J\tau_b'} \\ K_\pi' &= (K + C_1 L)_{J\tau=J\tau_b'-J\tau_a'} \end{aligned}\right\} \tag{41}$$

equations (37) and (38) may be simplified to

$$P_a = -AB\,\frac{\delta_n}{J^2}\cdot\zeta_f\cdot\Pi_a' + Aa_2\,\zeta_f\,(1-\varphi) \tag{42}$$

$$P_b = -AB\,\frac{\delta_n}{J^2}\,\zeta_f\,(\Pi_b' - \varphi\,K_\pi') + Aa_2\,\zeta_f\,(1-\varphi) \tag{43}$$

A5.3 Lateral accelerations (see Fig.4)

(i) at the C.G. of the aircraft

The critical values of the significant local maxima, n_{sa} and n_{sb}, are obtained when the recovery is initiated immediately the sideslip reaches a maximum in Stage II, i.e. when $d\beta/d\tau = 0$ so that $J\tau_c = J\tau_a$. Then (cf equation (25))

$$n_{sa} = -E\left(\bar{y}_v \frac{\delta_n}{J^2} \zeta_f \Pi_a - y_\zeta \zeta_f (1 - \varphi)\right) \tag{44}$$

and

$$n_{sb} = -E\left(\bar{y}_v \frac{\delta_n}{J^2} \zeta_f (\Pi_a - \varphi K_\pi) - y_\zeta \zeta_f (1 - \varphi)\right) \tag{45}$$

(ii) at the tail, due to angular acceleration in yaw

Here the critical values of the significant local maxima, $n_{\ell a} + n_{\ell b}$, are obtained when the recovery is initiated immediately a mathematical maximum of n_ℓ is reached in Stage II, i.e. when $dn_\ell/d\tau = 0$, say when $J\tau_c = J\tau''_a$. Then (cf equation (26))

$$n_{\ell a} = \frac{E}{\mu_3} \cdot \frac{\delta_n}{J^2} \cdot \left(\frac{\zeta_f}{J\tau_f}\right)\left(J^2 (L - L_f - k) + J\bar{y}_v (K - K_f)\right)_{J\tau = J\tau''_a} \tag{46}$$

and

$$n_{\ell b} = \frac{E}{\mu_3} \cdot \frac{\delta_n}{J^2} \cdot \left(\frac{\zeta_f}{J\tau_f}\right)\left(J^2 (L - L_f - k(H_c - \frac{R}{J} L_c)) + J\bar{y}_v (K - K_f - k L_c)\right)_{J\tau = J\tau''_b} \tag{47}$$

and if

$$\left.\begin{aligned}
\Lambda_a &= \frac{1}{J\tau_f}\left((L - L_f) + \frac{\bar{y}_v}{J} (K - K_f)\right)_{J\tau = J\tau''_a} \\
\Lambda_b &= \frac{1}{J\tau_f}\left((L - L_f) + \frac{\bar{y}_v}{J} (K - K_f)\right)_{J\tau = J\tau''_b} \\
\Lambda_c &= \left(k\left(H - \frac{R}{J} L\right) + \frac{y_v}{J} \cdot k \cdot L\right)_{J\tau = J\tau''_b - J\tau''_a}
\end{aligned}\right\} \tag{48}$$

equations (46) and (47) become

$$n_{\ell a} = \frac{E}{\mu_3} \cdot \delta_n \zeta_f \left(\Lambda_a - \varphi\right) \tag{49}$$

$$n_{\ell b} = \frac{E}{\mu_3} \cdot \delta_n \zeta_f \cdot \left(\Lambda_b - \varphi \Lambda_o\right) \tag{50}$$

A6 Simplifications

Calculations indicate that it may be assumed without significant loss of accuracy that

(i) the first critical maxima of β, P, n_s and n_ℓ occur at the same instant i.e. $J\tau_a = J\tau'_a = J\tau''_a$ and $n_{ta} = n_{sa} + n_{\ell b}$

(ii) the other critical maxima of these quantities occur at an interval $J\tau = \Pi$ after the first maxima i.e. $J\tau_b = J\tau_a + \pi$ etc. and $n_{tb} = n_{sb} + n_{\ell b}$

(iii) the functions Π_a etc. are numerically equal to the functions Π'_a etc.

(iv)

$$\left.\begin{aligned} \Lambda_a &= \frac{1}{J\tau_f}\left(L - L_f\right)_{J\tau = J\tau'''_a} \\ \Lambda_b &= \frac{1}{J\tau_f}\left(L - L_f\right)_{J\tau = J\tau'''_a + \pi} \\ \Lambda_o &= \left(H - \frac{R}{J} L\right)_{J\tau = \pi} \end{aligned}\right\} \tag{51}$$

where $J\tau'''_a$ is the first root beyond $J\tau_f$ of the equation

$$J\tau = \tan^{-1}\left\{\frac{-\frac{R}{J}\bar{L}_f - \left(\bar{H}_f - 1\right)}{\bar{L}_f - \frac{R}{J}\left(\bar{H}_f - 1\right)}\right\} \tag{52}$$

Thus, in practice, the estimation of the critical maxima of the various quantities (see equations (34) and (35), (42) and (43), (44) and (45) and (49) and (50)) only necessitates the estimation of the appropriate values of Π_a, Π_b and K_π (from equations (33) or (41) and Λ_a, Λ_b and Λ_o (from equation (49)). This work may be simplified by the use of charts, see Figs.5 and 6. The information contained in Fig.5 has been obtained from equations (36) and (41) with $C_1 = 0.3$; these particular equations have been used because Π'_a, Π'_b etc. are in fact up to 2% greater than the corresponding Π_a, Π_b etc. and the results so obtained lead to slightly conservative values of the maxima of β and n_s, and of P also if the actual value of C_1 is less than 0.3 (0.3 is the expected upper limit to C_1). The times of occurrence of the critical maxima, if required, may be estimated from Fig.7.

A7 Estimation of the amount of rudder displacement in the "check" stage

The stalling torque of the servomotor is usually known, but the external forces on the rudder (hinge moments) depend on the response of the aircraft, which, in turn, depends upon the magnitude of the rudder displacement. Thus, to calculate exactly the amount of displacement to stall the servomotor, ζ_f, a process of "trial and error" would have to be adopted. To ease this labour, two simplified methods are suggested for obtaining a good approximation to ζ_f

(i) from asymptotic conditions, assuming the recovery action is not taken.

The general expression for the hinge moment coefficient of the rudder in a yawing manoeuvre is (cf Ref.3)

$$C_h = -b_1\beta + b_2\zeta \tag{53}$$

The asymptotic conditions depend solely on the asymptotic rudder displacement and therefore equations (5) or (8) may be used to determine the asymptotic value of β. It is, assuming that the corresponding rudder displacement is ζ_f,

$$\beta = \frac{\delta_n}{J^2} \cdot K_a \zeta_f \tag{54}$$

and the asymptotic value of C_h is

$$C_h = \left(b_2 - \frac{\delta_n}{J^2} \cdot K_a b_1\right) \zeta_f \tag{55}$$

If this hinge moment coefficient is equated to the servomotor stalling torque then

$$\zeta_f = \frac{C_{h_s}}{\left(b_2 - \frac{\delta_n}{J^2} \cdot K_a b_1\right)} \tag{56}$$

(ii) from conditions arising from instantaneous rudder displacement.

With instantaneous movement of the rudder of magnitude ζ_f, the response in β see equation (8), is initially zero, and the corresponding hinge moment coefficient is

$$C_h = b_2 \zeta_f \tag{57}$$

and if this hinge moment is equated to the servomotor stalling torque then

$$\zeta_f = \frac{C_{h_s}}{b_2} \tag{58}$$

Method (i) is conservative when b_1 is positive, and method (ii) is conservative when b_1 is negative. It is suggested that when determining ζ_f, the sign of b_1 should first be examined. The conservative value of ζ_f (i.e. one larger than that occurring in practice) may then be estimated.

A8 Estimation of the amount of rudder movement in the recovery

The rudder hinge moment at the instant of recovery due to the runaway and check movements of the rudder is, (cf equation (53))

$$C_h = -b_1 \beta_a + b_2 \zeta_f \tag{59}$$

and the associated hinge moment due to the instantaneous recovery movement, ζ_c, is, since the response in β due to the recovery movement is initially zero,

$$C_h = b_2 \zeta_c \tag{60}$$

Thus, if, when the automatic pilot is disengaged, the pilot does not touch the rudder pedals, the rudder will be moved under the influence of the out of balance hinge moments until equilibrium is reached, i.e. till

$$-b_1 \beta_a + b_2 \zeta_f + b_2 \zeta_c = 0$$

or

$$\zeta_c = -\zeta_f + \frac{b_1}{b_2} \beta_a \tag{61}$$

Equation (61) may be considered as a condition to fix a lower limit to the amount of rudder displacement to be expected during the recovery since any action that the pilot might make would invariably be in the sense to move the rudder back still further; just how much farther is difficult to assess, and in any case is outside the scope of this paper.

It should be noted that if b_1 and b_2 are negative (in practice b_2 at least must be) the value of ζ_c from equation (61) will be less than ζ_f, whilst if b_1 is positive the rudder will be moved back beyond its neutral position before the resultant hinge moment drops to zero.

APPENDIX II

Computational procedure

A1 Introduction

A1.1 This Appendix contains details of the procedure for calculating the maximum angle of sideslip, maximum fin-and-rudder load and the maximum lateral acceleration, both at the C.G. of the aircraft and at the tail, that ensue from a failure in the rudder channel of an automatic pilot and the subsequent recovery.

A1.2 A list of the numerical data required is given in para.2 and particulars of the preliminary calculations are given in para.3. The formulae necessary for the calculation of the various maxima are given in para.4.

A1.3 Two formulae are given for each of the quantities; they refer to maxima that arise at different stages of the failure and subsequent recovery. Just which of the two maxima is critical in a practical case depends on the characteristics of the aircraft and the automatic pilot, and it is therefore necessary to consider both formulae to determine the absolute or critical maximum of the particular quantity in question.

A1.4 The formulae contain the functions Π_a, Π_b, K_π, Λ_a, Λ_b and Λ_c, and the numerical values of these functions may be obtained from Figs.5 and 6. The times of occurrence of the maxima may be obtained from Fig.7. Alternatively the numerical values of the functions may be obtained from the set of formulae in para.5.

A1.5 A numerical example illustrating the procedure is given in para.6. The data for this example are taken from Table I.

A2 List of data required (cf List of Symbols)

a_1	n_v	C_{h_s}
a_2	n_r	$\bar{\zeta}$ (radians)
b	y_v	ζ_f (radians)
b_1	S (ft)^2	ζ_c (radians)
$g = 32.2$ ft/sec^2	S'' (ft)^2	$\left(\frac{d\zeta}{dt}\right)_f$ (radians/sec)
k_c (ft)	V (ft/sec)	
ℓ (ft)	W (lb)	
ℓ_R (ft)	ρ (slugs/ft^3)	

Note: ζ_f and $(d\zeta/dt)_f$ are negative when the rudder is displaced to starboard; ζ_c is of opposite sign to ζ_f; C_{h_s} is positive when the rudder is displaced to starboard.

A3 Basic quantities to be evaluated (cf List of Symbols)

A3.1

$$\mu_2 = \frac{2W}{g\rho Sb} = \frac{2\ell}{b} \cdot \mu_3 \qquad \overline{V}_R = \frac{S''\ell_R}{Sb}$$

$$\mu_3 = \frac{W}{g\rho S\ell} = \frac{b}{2\ell} \mu_2 \qquad \omega_n = \frac{1}{i_c} \mu_2 n_v$$

$$\hat{t} = \frac{W}{g\rho SV} = \frac{\ell}{V} \mu_3 \qquad \delta_n = \frac{1}{i_c} \mu_2 \overline{V}_R a_2$$

$$A = \tfrac{1}{2}\rho V^2 S'' \qquad \nu_n = -\frac{1}{i_c} n_r$$

$$B = (1 + \overline{y}_v/\mu_3)a_1 \qquad \overline{y}_v = -y_v$$

$$E = 2 \cdot \frac{\tfrac{1}{2}\rho V^2}{W/S} \qquad y_\zeta = \tfrac{1}{2} \frac{S''}{S} a_2$$

$$i_c = \frac{4k_c^2}{b^2} \qquad R = \tfrac{1}{2}(\nu_n + \overline{y}_v)$$

$$K_a = \frac{1}{\left(\frac{R}{J}\right)^2 + 1} \qquad J = \sqrt{\omega_n - \tfrac{1}{4}(\nu_n - \overline{y}_v)^2}$$

The relationship between time (t) and $J\tau$ is $t = \frac{\hat{t}}{J} (J\tau)$.

A3.2

(i) ζ_f If b_1 is negative then $\zeta_f = \overline{\zeta}_f$ or Ch_s/b_2 radians, whichever of the two is less. But if b_1 is positive then $\zeta_f = \overline{\zeta}_f$ or $Ch_s/(b_2 - \delta_n K_a b_1)$ radians, whichever of the two is less.

(ii) φ and $J\tau_f$

$$\varphi = -\left(\frac{\zeta_c}{\zeta_f}\right) \qquad J\tau_f = \frac{J\zeta_f}{\hat{t}(d\zeta/dt)_f}$$

A3.3 In addition Π_a, Π_b, K_π, Λ_a, Λ_b and Λ_o must be evaluated. Here, either Figs.5 and 6 or the appropriate formulae of para.5 may be used. In most cases the graphs will be sufficient. Further, if the times of occurrence of the maxima are required, they may be found from Fig.7. The first set of maxima occur after $J\tau_a' \hat{t}/J$ secs and the second set after $(\pi + J\tau_a') \hat{t}/J$ secs where $J\tau_a'$ is expressed in radians.

A4 Formulae

Quantity	Formulae for maxima	Associated graphs	Approx. time of occurrence of maxima	Associated graph
Angle of Sideslip	$\beta_a = \frac{\delta_n}{J^2} \cdot \zeta_f \cdot \Pi_a$	Fig. 5a	$J\tau = J\tau'_a$	Fig. 7
	$\beta_b = \frac{\delta_n}{J^2} \cdot \zeta_f \cdot (\Pi_b - \varphi K_\pi)$	Figs. 5a and 5b	$J\tau = \pi + J\tau'_a$	"
Fin-and-rudder load	$P_a = -AB \cdot \frac{\delta_n}{J^2} \cdot \zeta_f \cdot \Pi_a + A a_2 \zeta_f (1 - \varphi)$	Fig. 5a	$J\tau = J\tau'_a$	"
	$P_b = -AB \cdot \frac{\delta_n}{J^2} \cdot \zeta_f \cdot (\Pi_b - \varphi K_\pi) + A a_2 (1 - \varphi)$	Figs. 5a and 5b	$J\tau = \pi + J\tau'_a$	"
Coeff. of lateral acceleration at the C.G.	$n_{sa} = -E\left(\bar{y}_v \cdot \frac{\delta_n}{J^2} \cdot \zeta_f \cdot \Pi_a - y_\zeta \zeta_f (1 - \varphi)\right)$	Fig. 5a	$J\tau = J\tau'_a$	"
	$n_{sb} = -E\left(\bar{y}_v \cdot \frac{\delta_n}{J^2} \cdot \zeta_f (\Pi_b - \varphi K_\pi) - y_\zeta \zeta_f (1 - \varphi)\right)$	Figs. 5a and 5b	$J\tau = \pi + J\tau'_a$	"
Coeff. of lateral acceleration at the tail due to acceleration in yaw	$n_{\ell a} = \frac{E}{\mu_3} \cdot \delta_n \cdot \zeta_f \cdot (\Lambda_a - \varphi)$	Fig. 6a	$J\tau = J\tau'_a$	"
	$n_{\ell b} = \frac{E}{\mu_3} \cdot \delta_n \cdot \zeta_f \cdot (\Lambda_b - \varphi \Lambda_o)$	Figs. 6a and 6b	$J\tau = \pi + J\tau'_a$	"
Coeff. of total lateral acceleration at the tail	$n_{ta} = n_{sa} + n_{\ell a}$	-	$J\tau = J\tau''_a$	"
	$n_{tb} = n_{sb} + n_{\ell b}$	-	$J\tau = J\tau'_a + \pi$	"

A5 Formulae for the functions Π_a etc.

(i)

$$\Pi_a = \frac{1}{J\tau_f}\left((G - G_f) + 0.3\,(K - K_f)\right)_{J\tau=J\tau'_a}$$

where $J\tau'_a$ is the first root beyond $J\tau_f$ of the equation

$$J\tau = \tan^{-1}\left\{\frac{Z\overline{L}_f - (\overline{H}_f - 1)}{\overline{L}_f + Z\,(\overline{H}_f - 1)}\right\}$$

and

$$\overline{L}_f = e^{\frac{R}{J}J\tau_f}\sin J\tau_f,\quad \overline{H}_f = e^{\frac{R}{J}J\tau_f}\cos J\tau_f \quad \text{and} \quad Z = \frac{R}{J} - \frac{0.3}{K_a}$$

(ii)

$$\Pi_b = \frac{1}{J\tau_f}\left((G - G_f) + 0.3\,(K - K_f)\right)_{J\tau=J\tau'_b}$$

where $J\tau'_b$ is the first root beyond $J\tau'_a$ of the equation

$$J\tau = \tan^{-1}\left\{\frac{Z\overline{L}_f - (\overline{H}_f - 1) + Z_1\,\overline{L}_a + Z_2\,\overline{H}_a}{\overline{L}_f + Z(H_f - 1) - Z_2\,\overline{L}_a + Z_1\,\overline{H}_a}\right\}$$

and

$$Z_1 = \frac{k}{K_a}\left(0.3\left(\frac{R}{J}\right) - 1\right) \qquad Z_2 = \frac{0.3\,k}{K_a}$$

$$\overline{H}_a = e^{\frac{R}{J}\cdot J\tau'_a}\cos J\tau'_a \quad \text{and} \quad \overline{L}_a = e^{\frac{R}{J}J\tau_a}\sin J\tau'_a$$

(iii)

$$K_\pi = (K + 0.3L)_{J\tau=J\tau'_b-J\tau'_a}$$

(iv)

$$\Lambda_a = \frac{1}{J\tau_f}\,(L - L_f)_{J\tau=J\tau'''_a}$$

where $J\tau_a'''$ is the first root beyond $J\tau_f$ of the equation

$$J\tau = \tan^{-1}\left\{\frac{-\frac{R}{J}\,\overline{L}_f - (\overline{H}_f - 1)}{\overline{L}_f - \frac{R}{J}\,(\overline{H}_f - 1)}\right\}$$

(v)

$$\Lambda_b = \frac{1}{J\tau_f}\,(L - L_f)_{J\tau=\pi+J\tau_a'''}$$

(vi)

$$\Lambda_o = \left(H - \frac{R}{J}\,L\right)_{J\tau=\pi}$$

Further details of these formulae are given in Appendix I.

A6 <u>Numerical example</u>

The details are given under the relevant paragraph headings of this Appendix.

<u>Para. A3</u>

A3.1 all the necessary data are contained in Table I.

A3.2 b_1 is negative, $\overline{\zeta} = 0.2093$ and $C_{h_s}/b_2 = -0.0513/-0.3 = 0.171$ rads so that $\zeta_f = 0.171$ rads.

$$\varphi = 1 \quad \text{and} \quad J\tau_f = \frac{4.293 \times 0.171}{1.34 \times 0.1745} = 3.142 \simeq \pi$$

A3.3 With $R/J = 0.093$ and $J\tau_f = \pi$ we obtain from Figs. 5 and 6

$$\Pi_a = 1.483 \qquad \Lambda_a = -0.482$$

$$\Pi_b = 0.623 \qquad \Lambda_b = 0.361$$

$$K_\pi = 1.763 \qquad \Lambda_o = -0.759$$

also, from Fig. 7, $J\tau_a' = 257^\circ$ so that the first set of maxima occur after $(257 \times 1.34/57.3 \times 4.293)$ secs, i.e. 1.4 secs. The second set of maxima occur after $(257/57.3 + \pi)\ 1.34/4.243$ secs or 2.38 secs.

Para. A4

Since

$$\frac{\delta_n}{J^2} \cdot \zeta_f = \frac{22.53}{4.293^2} \times 0.171 = 0.2093$$

$$\beta_a = 0.2093 \times 1.483 \text{ rads} = 0.31 \text{ rads}$$

$$\beta_b = 0.2093 \times (0.623 - 1.763) \text{ rads} = -0.235 \text{ rads}$$

$$P_a = -6400 \times 2.517 \times 0.2093 \times 1.483 \text{ lb} = -5000 \text{ lb}$$

$$P_b = -6400 \times 2.517 \times 0.2093 \times (0.623 - 1.763) \text{ lb} = 3750 \text{ lb}$$

$$n_{sa} = -11.8 \times 0.23 \times 0.2093 \times 1.483 = -0.84$$

$$n_{sb} = -11.8 \times 0.23 \times 0.2093 (0.623 - 1.763) = 0.62$$

$$n_{\ell a} = \frac{11.8}{29.44} \times 22.53 \times 0.171 \times (-0.482 - 1) = -2.29$$

$$n_{\ell b} = \frac{11.8}{29.44} \times 22.53 \times 0.171 \times (0.361 + 0.759) = 1.73$$

$$n_{ta} = -0.84 - 2.29 = -3.13$$

$$n_{tb} = 0.62 + 1.73 = 2.35$$

Thus, in the present example, the critical conditions are

$$\beta = 0.31 \text{ rads}$$

$$P = -5000 \text{ lb}$$

$$n_s = -0.84$$

$$n_t = -3.13$$

they arise 1.4 secs after failure of the automatic pilot.

TABLE I

μ_3	=	29.44	b_1	=	-0.1 per radian
$\hat{t}$	=	1.34 secs	b_2	=	-0.3 per radian
A	=	6400 lb	$\bar{\zeta}$	=	12°
E	=	11.8		=	0.2093 radians
J	=	4.293	ζ_f	=	9.81° (See Appendix II Para.6)
$\frac{R}{J}$	=	0.093		=	0.171 radians (See Appendix II Para.6)
δ_n	=	22.53	ζ_c	=	$-\zeta_f$
$\bar{y}_v$	=	0.23	$\left(\frac{d\zeta}{dt}\right)_f$	=	10°/sec
y_ζ	=	0.067		=	0.1745 rads/sec
a_2	=	1.8 per radian	C_{h_s}	=	-0.0513

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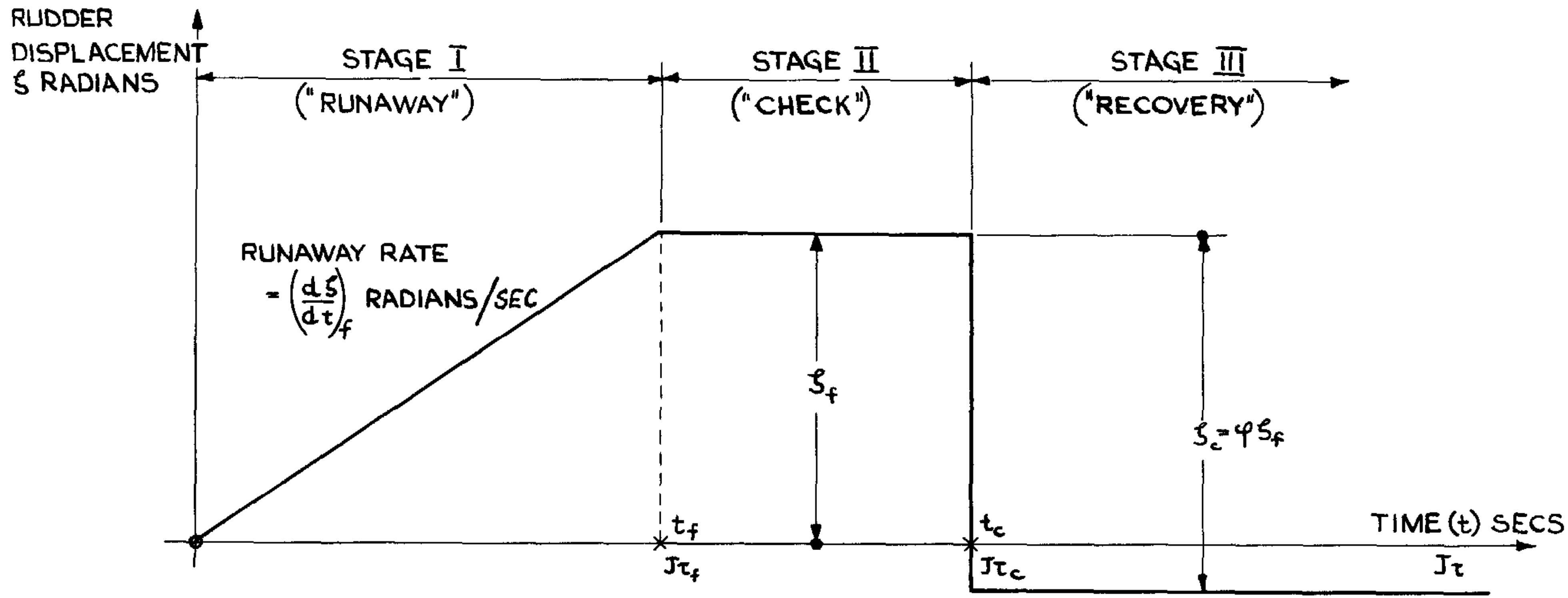

FIG. 1 ASSUMED RUDDER TIME-HISTORY

FIG. 1

FIG. 2

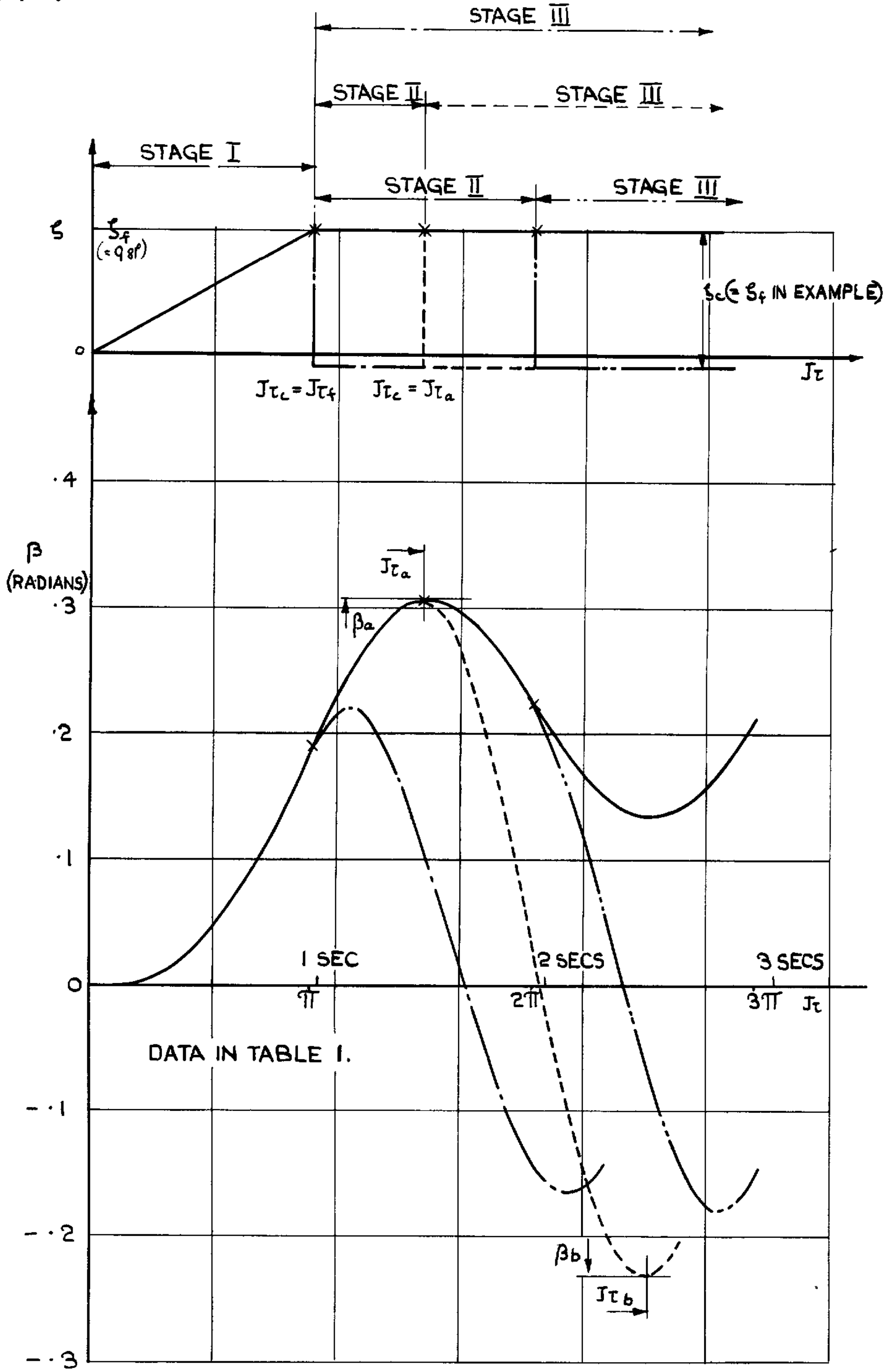

FIG. 2. RESPONSE IN ANGLE OF SIDESLIP (VARIOUS RECOVERY TIMES)

FIG 3.

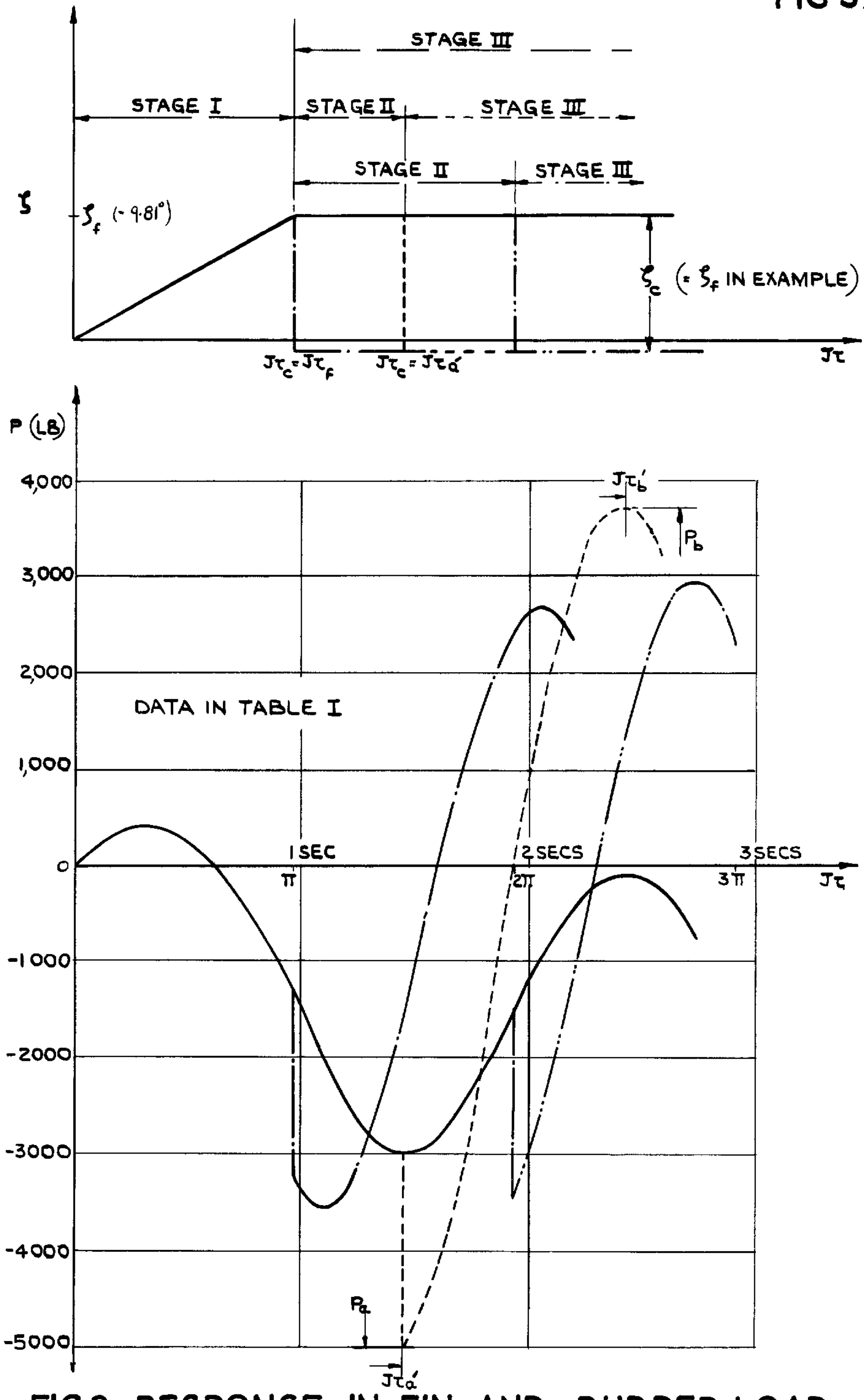

FIG 3. RESPONSE IN FIN-AND-RUDDER LOAD.
(VARIOUS RECOVERY TIMES)

FIG 4.

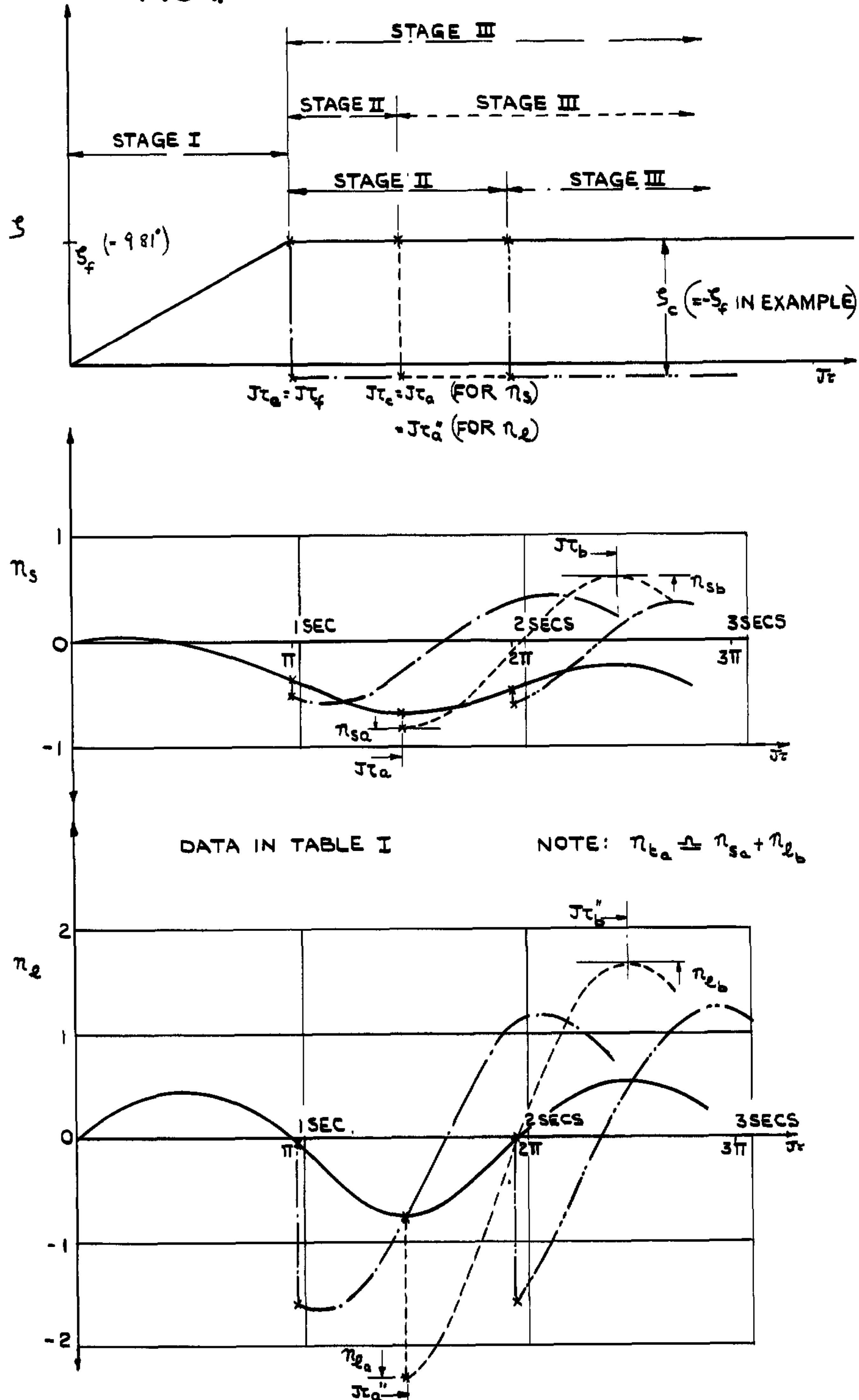

FIG 4. RESPONSE IN LATERAL ACCELERATION
(VARIOUS RECOVERY TIMES.)

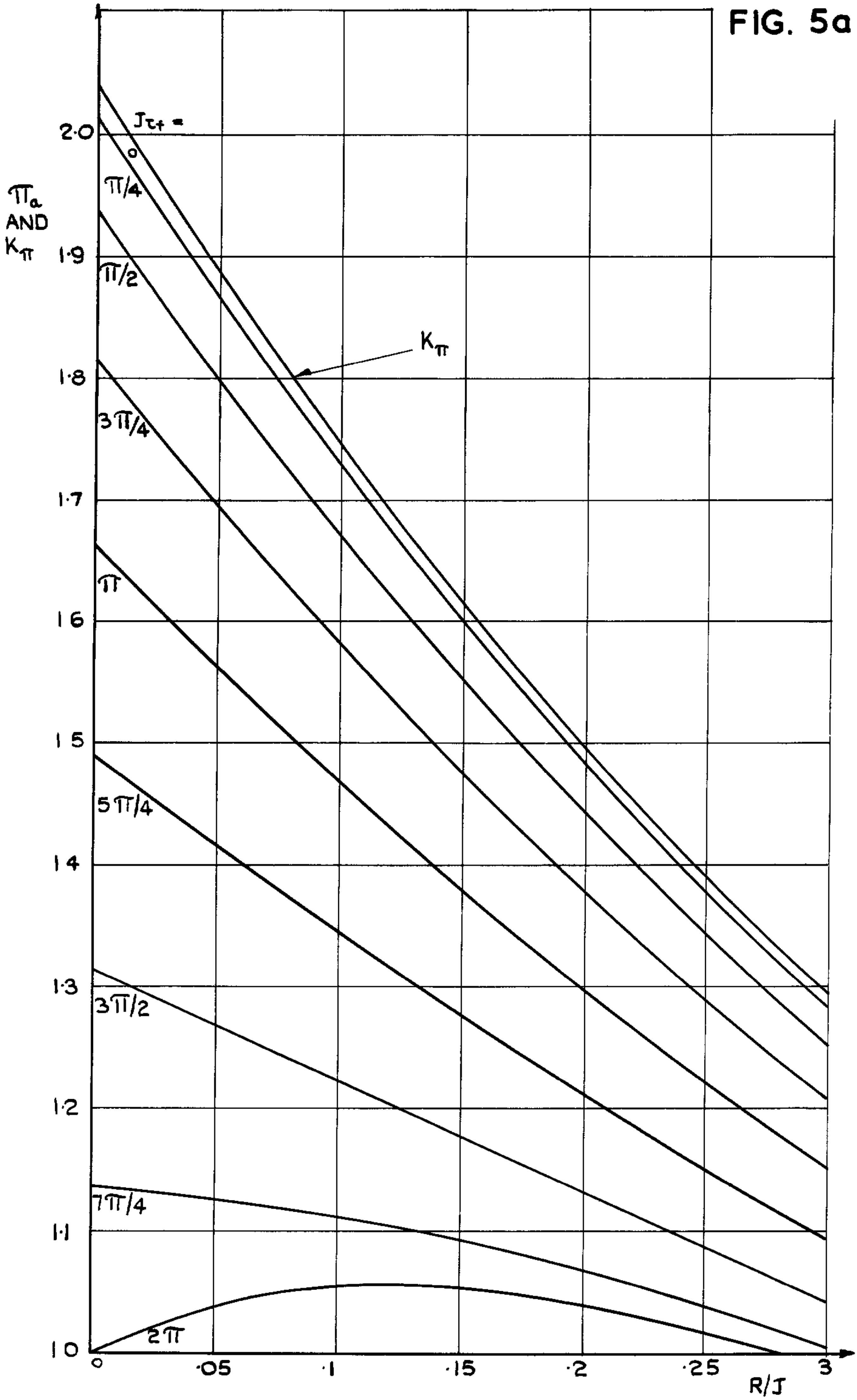

FIG. 5a. CHART SHOWING FUNCTIONS Π_a, Π_b AND K_π

FIG. 5b

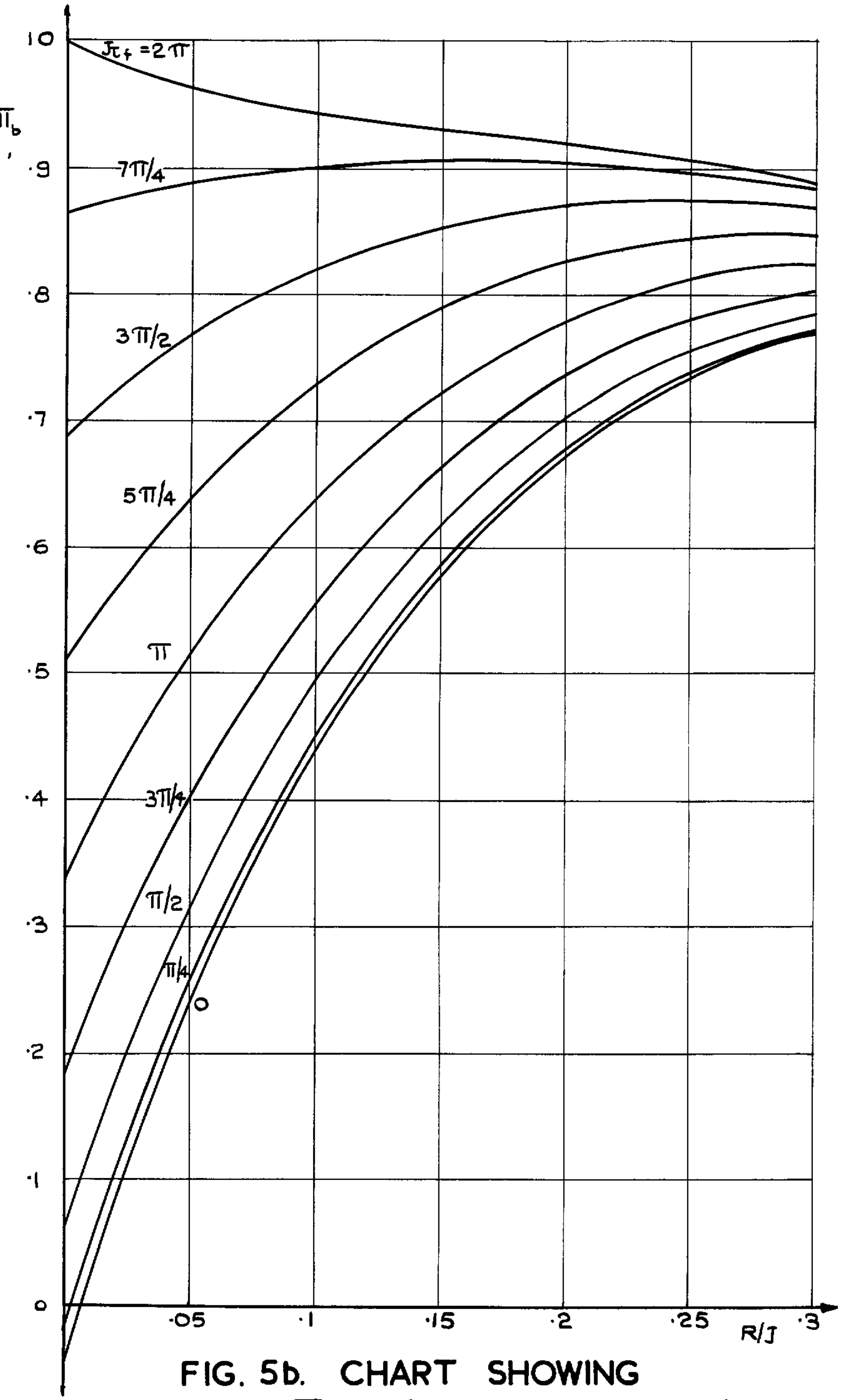

FIG. 5b. CHART SHOWING FUNCTIONS Π_a, Π_b AND K_π (CONT)

FIG .6a

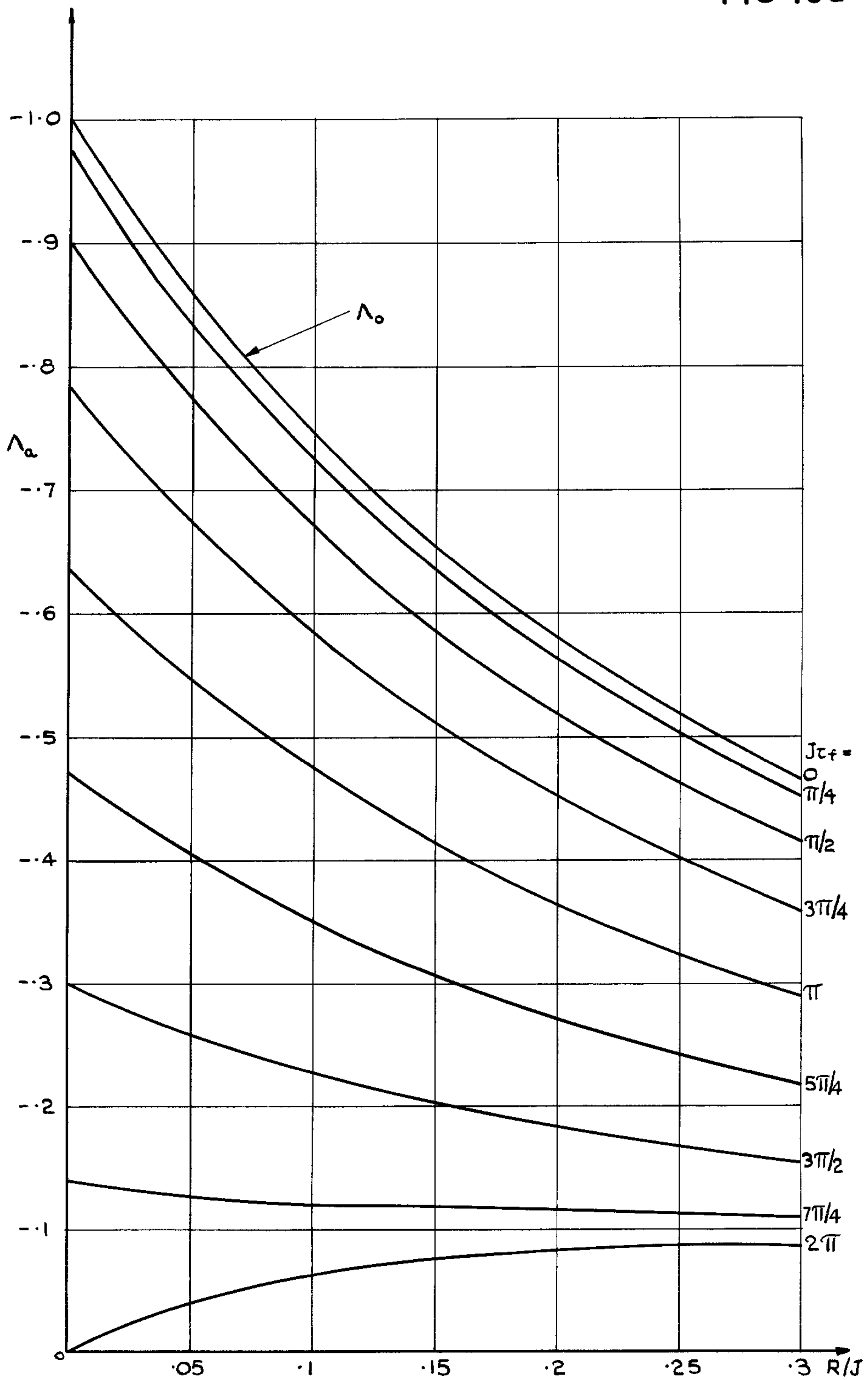

FIG. 6a CHART SHOWING FUNCTIONS Λ_a, Λ_b AND Λ_o.

FIG. 6b

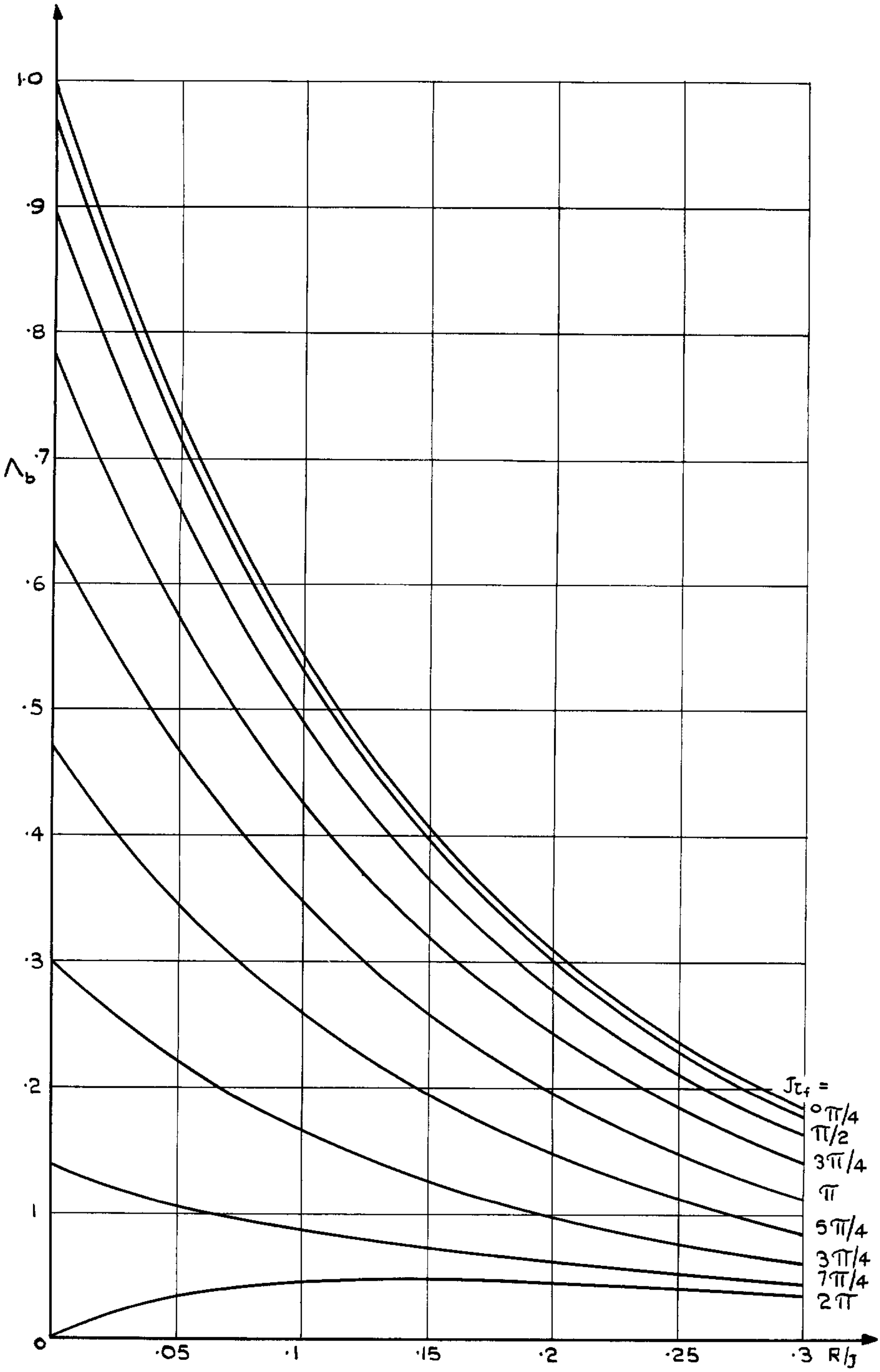

FIG. 6b CHART SHOWING FUNCTIONS Λ_a, Λ_b AND Λ_o (CONT)

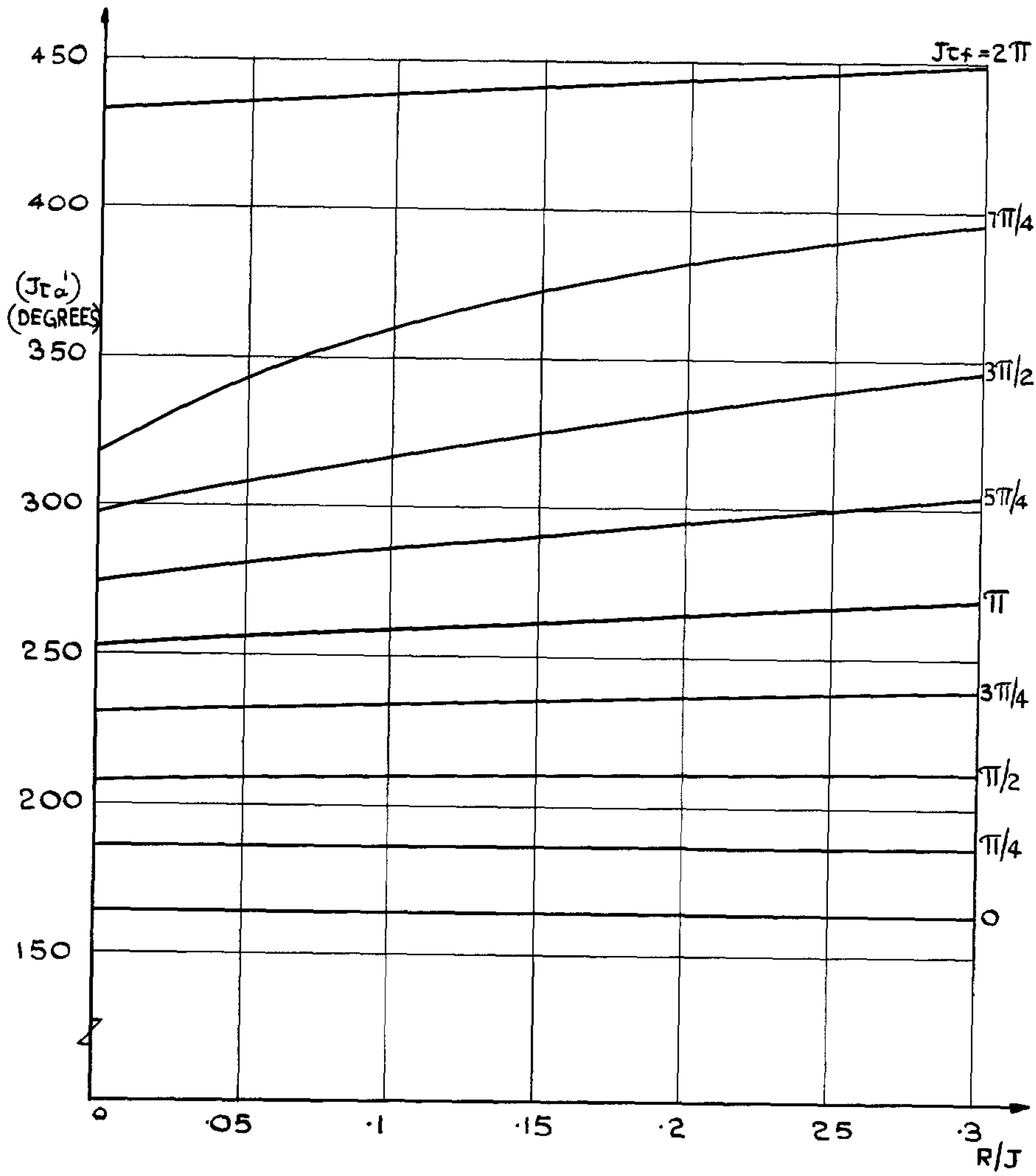

FIG. 7. CHART SHOWING TIME OF OCCURRENCE OF THE MAXIMA

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