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CURRENT PAPERS

Transitional Friction Effects in Powered Controls with Particular Reference to Hydraulic Jacks

By

F. HOLOUBEK, A.F.R.Ae.S.

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F. Holoubek, A.F.R.Ae.S.

SUMMARY

The motion of a system in which power is transmitted from a constant rate source into an output component of finite mass through an elastic medium, in which the output element is restrained by friction, is analysed and discussed. It is shown that the result is a quasi-oscillatory sustained motion which includes a period of rest between consecutive cycles. It is submitted that this phenomenon is the basis of the so called "judder" encountered in servo-mechanisms, control circuits etc. Means of mitigating the severity of the "judder" are suggested.

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LIST OF SYMBOLS

x_1, x_{1_o}, x_1', x_1''	steady rate input displacements	ft
x_2, x_2', $x_{2_{v max}}$	output displacements	ft
Δx	output lag, spring compression	ft
t, t_o, t', t'', $t_{v_{2max}}$	times, periods	sec
T	period of cycle	sec
f	frequency	$\frac{1}{sec}$
F_s	spring force	lb
F_L	friction per unit length of periphery of a packing	lb/in
r	kinetic friction	lb
R	static friction	lb
μ	viscosity	c.p.
p	working pressure lb/in^2 (also integrating factor in Laplace transform e^{-pt})	
p_i	inlet pressure	lb/in^2
p_ℓ	exhaust pressure	lb/in^2
p_o	interference pressure	lb/in^2

v		rubbing velocity ft/sec
v_1		equivalent steady input rate ft/sec
v_2		absolute piston velocity ft/sec
D, d		jack cylinder bore and rod diameter in.
b		width of packing in.
c	=	$\frac{\rho_o}{p}$
h		film thickness in.
k		spring constant lb/ft
m		mass $\frac{\text{lb sec}^2}{\text{ft}}$
ρ		velocity ratio

1 Introduction

A mechanical device comprising a mass and a spring constitutes an oscillatory system. Any form of damping, friction or velocity damping generally causes decay of the existing oscillation, unless energy is fed into the system to sustain the amplitude. There is, however, one instance of motion in which friction is the cause and condition of sustained oscillatory behaviour. It is the case of incremental motion of a journal in a bearing, a rod in a sleeve, a pin in a hinge and others. This case is of particular interest in the application of rubber seals in the jacks of hydraulically powered flying controls. It has been observed in certain power installations that, while operated on the ground, within a range of input rates the piston rod and consequently the control surface moved in a series of increments, a phenomenon sometimes called "judder", giving a visual and audible impression of a violent oscillation superimposed on the mean linear displacement.

Let, for the purpose of analysis, the moment of inertia of the rotating control surface with its mass balance be replaced by a dynamically equivalent mass concentrated in the piston of the operating jack. Further, let the combined elasticity of the system be measured or estimated and assumed to be made up of the compressibility of the fluid and the elasticity of the envelope (expansion of the jack cylinder and pipes). Finally, let it be imagined that a constant speed hydraulic pump begins delivering pressure oil at a rate corresponding to a mean steady piston velocity v_1. This assumption would strictly apply only in the case of pure rate control without a "follow-up" mechanism, as the follow-up valve would impress its characteristic on the volume delivered into the jack. As it happens in some cases of valve control, the effects of friction are felt most distinctly in the range of medium speeds v_1, when the valve is only partly open and its fluctuation renders the analysis rather complicated. It is assumed, however, that the present simple theory does apply in the case of a long stroke control valve, where a small fluctuation of the fully open valve does not appreciably alter the rate of flow.

2 General analytical method

2.1 Friction

Considerable uncertainty prevailed in the assessment of the value of friction coefficient of rubber seals as used in hydraulic jacks, prediction being made difficult by the seemingly erratic behaviour of rubber in the lubricated bore of the jack cylinder. Among the factors influencing the friction coefficient are material and degree of finish of the bore, composition and hardness of the rubber sealing ring, velocity of rubbing, operating pressure, type of hydraulic fluid, initial contact pressure (interference fit of the seal) and a number of others. Experience from hydraulic undercarriage struts indicates that some 5-10% of thrust is dissipated in the friction of the glands. White and Denny in their recent work (Reference No.1) made a welcome contribution towards understanding the sealing mechanism of flexible packings. Working on the theory advanced by Archbut and Deely and Michell's theory of flow between inclined surfaces, they evolved a formula connecting the main factors in the form of a general equation

$$F_L = \sqrt{\frac{p \mu V b}{2}} \quad (2.8 + 2c)$$

where F_L = friction per unit length of circumference of the seal, p = working pressure of the fluid, μ = fluid viscosity, V = speed of rubbing, b = axial width of the packing ring, $c = \frac{p_o}{p}$ and p_o = initial contact pressure (interference fit). In the authors' own admission this formula may be erring on the low side.

A point of particular importance, however, is the experimental evidence of an extremely steep rise of the friction coefficient at rubbing speeds below a certain critical value. This critical velocity is defined as the one at which a permanent film of fluid (lubricant) is established which separates the two rubbing surfaces. Above that velocity the White-Denny formula applies and the thickness of the film h can be calculated from the equation

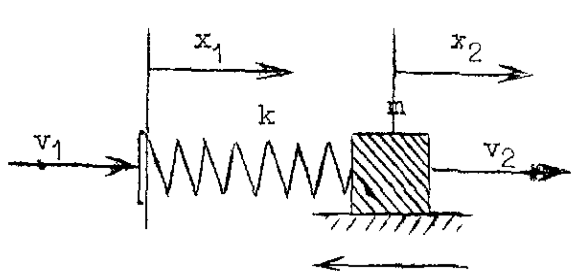
$$h = \sqrt{\frac{\mu v b}{p(16.2 + 380)}}$$

At speeds below the critical, however, the film of lubricant begins to break down and, probably due to the extreme thinness and discontinuity of the film, the surface adhesive forces come into play, resulting in a steep rise in the rubbing resistance. It is, of course, a matter of common knowledge that the static, or breakaway friction is usually somewhat higher than the kinetic friction, which is generally true even of non-lubricated surfaces, but the rise in frictional resistance shown by White and Denny (see Reference No.1, pp.49 and 56, reproduced in figure 2 of this Report) is so unexpectedly abrupt and large at speeds just below critical, that it has, in some cases, the character of discontinuity or step change. Thus values of friction under static (boundary) conditions of 5 to 10 times those of film friction seem to be quite commonplace and, depending on the combination of type of rubber and lubricant, may well be exceeded. This is an extremely interesting property of lubricated seals and its implications are discussed in the following paragraphs.

2.2 Equation of motion

One complete cycle of motion of the system described in the first paragraph will be found to consist of three phases.

First Phase. Let it be first assumed that the system be initially at a standstill, i.e. the valve is closed and there is no flow in or out of the jack. At time t = 0 the valve opens fully and the pump begins to deliver fluid into the jack at a constant rate corresponding to an equivalent piston velocity v_1. Simultaneously pressure is applied to the seals giving rise to a high static friction R, which resists the piston moving. As, however, the pump is delivering all the time at a constant rate pressure in the system rises and, until the piston "breaks away" and starts moving, the excess volume of fluid is accommodated in the expanding cylinder and pipelines. In the symbolised representation



when $t \leq 0$ $x_1 = x_2 = 0$ $v_2 = 0$

$0 < t \leq t_o$ $x_1 = v_1 t$ $x_2 = 0$

No movement of the mass m will take place until the spring force kx_1 reaches the level of R, when the seal will "break away". This condition is expressed as

$$v_1 \, t_o \, k = R \qquad \text{or}$$

$$x_{1_o} = v_1 t_o = \frac{R}{k} \tag{1}$$

and

$$t_o = \frac{R}{kv_1} \tag{2}$$

Second Phase. Almost immediately after the "break away" the friction force R drops to a fractional value r and will, for simplicity of treatment be assumed constant as long as the motion persists. This assumption of constant friction is not entirely correct. The absolute piston velocity v_2 varies, as will be seen, from 0 to approximately $2v_1$ within one cycle and the friction force, according to White and Denny varies as the square root of velocity. Thus, if calculation of friction is based on the mean velocity v_1 the instantaneous error will be within ±40%. But the overall effect of this variation within one cycle and in particular its effect on the frequency will be found negligible.

Measuring time t from the instant of the "break away" the spring force at any instant can be expressed as

$$\begin{aligned} F_s &= k\Delta x \\ &= k(x_1 - x_2) \\ &= k(x_{1_o} + v_1 t - x_2) \end{aligned} \tag{3}$$

and inserting (1) for x_{1_o}

$$F_s = k\left(\frac{R}{k} + v_1 t - x_2\right) \tag{4}$$

Now the equation of motion can be written

$$-m\frac{d^2x_2}{dt^2} - r + F_s = 0 \tag{5}$$

Using (4) for F_s and arranging the terms in the way suitable for operational solution

$$\phi(D) = \psi(t)$$

the equation of motion becomes

$$\frac{d^2x_2}{dt^2} + \frac{k}{m} x_2 = \frac{k}{m} v_1 t + \frac{R-r}{m} \tag{6}$$

Transforming (6) (Laplace)

$$\left(p^2 + \frac{k}{m}\right)\int_0^\infty \ell^{-pt} x_2 \, dt = \frac{1}{m}\int_0^\infty (kv_1 t + R-r) \, \ell^{-pt} \, dt \tag{7}$$

complete solution of (6) is obtained

$$x_2 = v_1 \left[t - \frac{1}{\sqrt{\frac{k}{m}}} \sin\sqrt{\frac{k}{m}}\, t\right] + \frac{R-r}{k}\left[1 - \cos\sqrt{\frac{k}{m}}\, t\right] \tag{8}$$

It can be seen from (8) that the motion is basically an oscillation about the mean datum $v_1 t$, somewhat assymetric due to the presence of friction. (See sketch page 9 and figures 3, 4, 5 and 7).

The duration of this phase t' is obtained from the condition of maximum amplitude by differentiating (8) with respect to time, equating to zero and solving for t, thus

$$t' = \frac{2\pi}{\sqrt{\frac{k}{m}}} - \frac{1}{\sqrt{\frac{k}{m}}} \sin^{-1} \frac{2(R-r)\ v_1\ m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1^2\ k\, m} \qquad \text{or}$$

$$= \frac{\pi}{\sqrt{\frac{k}{m}}} + \frac{1}{\sqrt{\frac{k}{m}}} \sin^{-1} \frac{2(R-r)\ v_1\ m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1^2\ k\, m} \tag{9}$$

depending on whether the maximum lies in the fourth or third quadrant (See Appendix I).

<u>Third Phase</u>. At the instant t' the piston velocity becomes zero and as a consequence the friction rises again from r to its static value R and the motion will cease for such a period of time as the spring force remains less than the friction $\pm R$. At the instant t' the amplitude x_2 was

$$x_2' = v_1 \left[t' - \frac{1}{\sqrt{\frac{k}{m}}} \sin\sqrt{\frac{k}{m}}\, t' \right] + \frac{R-r}{k} \left[1 - \cos\sqrt{\frac{k}{m}}\, t' \right] \tag{10}$$

and at the same time x_1 was

$$x_1' = x_{1_o} + v_1 t' = \frac{R}{k} + v_1 t' \tag{11}$$

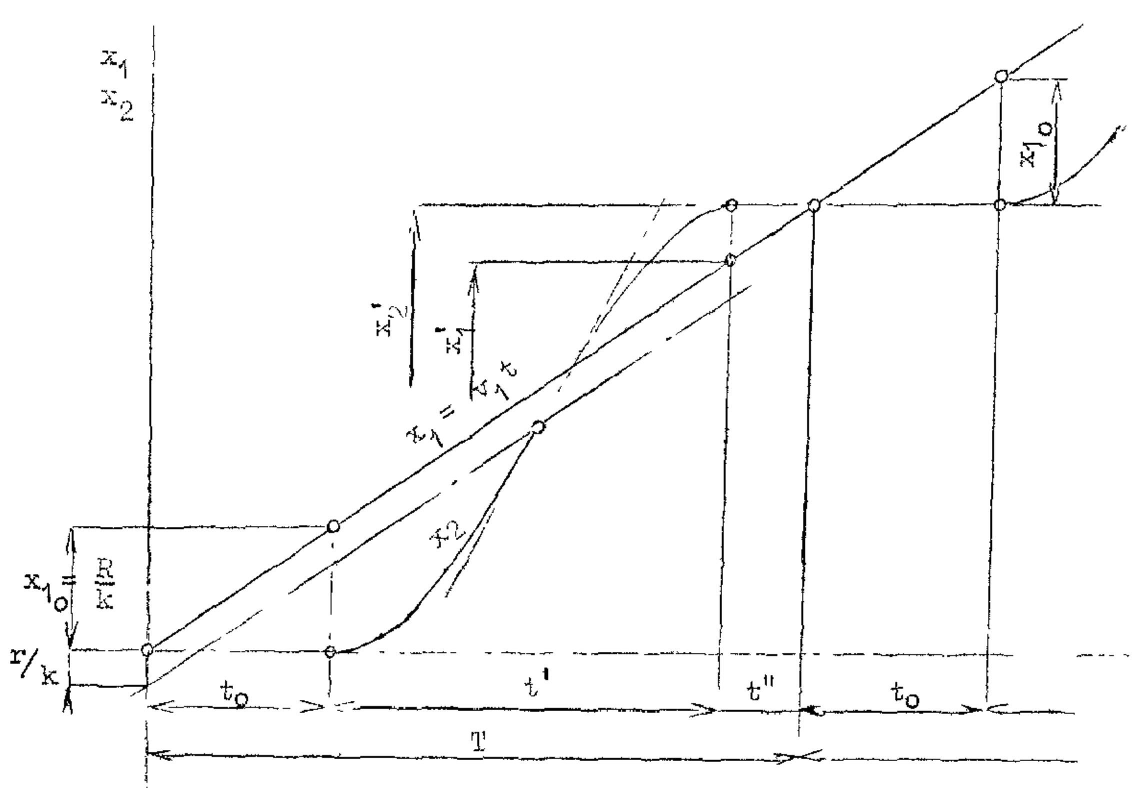
The time t'' (measured from the instant t') necessary to complete the cycle, i.e. when again $x_2 = x_1$ is calculated from

$$t'' = \frac{\Delta x}{v_1} = \frac{x_2' - x_1'}{v_1} \tag{12}$$

Inserting expressions (10) and (11) for x_2' and x_1' in (12) a simple expression is obtained for t'', thus

$$t'' = \frac{R - 2r}{kv_1} \tag{13}$$

irrespective of the quadrant in which the maximum of x_2 lies.



At the instant t'' the initial conditions $\left(x_1 = x_2,\ \frac{dx_1}{dt} = v_1,\ v_2 = 0\right)$ have been reproduced and a new identical cycle will begin.

The total period of the cycle will be the sum of the intervals of the three phases, thus

$$T = t_o + t' + t'' \tag{14}$$

where the right hand side is the sum of equations (2), (9) and (13). Because of the duality of (9) the total time T is either

$$T = \frac{2\pi}{\sqrt{\frac{k}{m}}} + \frac{2(R-r)}{kv_1} - \frac{1}{\sqrt{\frac{k}{m}}} \sin^{-1} \frac{2(R-r)\ v_1\ m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1^2\ k\,m} \tag{15a}$$

or

$$T = \frac{\pi}{\sqrt{\frac{k}{m}}} + \frac{2(R-r)}{kv_1} + \frac{1}{\sqrt{\frac{k}{m}}} \sin^{-1} \frac{2(R-r)\ v_1\ m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1^2\ k\,m} \tag{15b}$$

Referring to (15a) the first term of the right hand side is clearly the natural period T_o of the same system without friction and it is apparent, and can be generally shown (for 15b above), that T is always greater than T_o, in other words, the addition of this type of friction reduces the natural frequency of a frictionless system.

2.3 Discussion of method. Graphical representation.

For the purpose of graphical analysis arbitrary values were introduced into equation (8), viz. $W = 1$ lb, $k = 10$ lb/ft, $R = 0.4$ (0.8, 1.6) lb, $r = 0.2$ lb and $v_1 = 1.0$ (0.5, 2.0) ft/sec. The results were tabulated and plotted (figures 3 and 4). First the steady input rate v_1 was varied ($v_1 = 0.5$, 1.0, 2.0 ft/sec). It can be seen (figure 3) that with increasing v_1 the period of rest between consecutive cycles (during which $\frac{dx_2}{dt} = 0$) which is characteristic of incremental (as distinct from true oscillatory) motion, decreases to a small fraction of the total period of the cycle. One possible inference from this fact is that above a certain value of v_1 the period of rest becomes so small that the film established during the preceding part of the cycle has no time to break down with the result that a somewhat constant low value of friction coefficient prevails throughout the cycle. The motion, however, always remains incremental, although with decreasing period of rest the incremental character of the cycle becomes less and less patent.

It is interesting to note that the total period T and its reciprocal, the frequency, is practically independent of v_1, with a scarcely perceptible tendency for the frequency to rise with v_1.

The maximum instantaneous amplitude relative to the steady input rate datum v_1t (displacement lag) increases with v_1, but the relative amplitude, that is the ratio of maximum instantaneous amplitude and the total displacement in one cycle (between two consecutive rests) falls off with growing v_1. It may be that the violence of oscillation of this type should be gauged by its relative maximum amplitude rather than by its absolute value, or, at any rate the psychological impression in the observer may be some function of the two combined, which, if accepted, may explain why the motion would "appear" most uneven within a certain medium range of steady input velocity v_1. Another possible argument is that while at low v_1 the maximum amplitude is small and not too significant, in the region of high v_1 in hydraulic circuits hydraulic damping forces come into play, proportional to v^N (N=1 to 2) which have not been considered in this treatise for reason of simplicity, tending to even out the oscillation. That leaves again a certain medium range of v_1 where the incremental motion appears relatively most violent.

Quite accidentally in the three cases considered (figure 3) the value of static friction was chosen twice that of kinetic friction, R=2r. This is a special case and it will have been already noticed that in this case t'' becomes zero and the cycle has only two phases.

In figure 4 a set of three cases is plotted in which the ratio R/r was varied (R/r = 2, 4, 8). The increase of the period of rest with R/r is very obvious, but again the total period of the cycle is little affected, there being a tendency for the frequency to drop with increasing R/r.

In figure 5 is shown the effect of stiffening of the transmitting medium, k. The result is twofold: the frequency of the cycle increases appreciably with k, and simultaneously the period of rest (t_o and t'') quickly diminishes. The latter effect leads to a situation similar to that encountered above with high v_1 and similar considerations apply here. Thus, in a given system, raising the stiffness of the transmitting medium is clearly an effective way of combating the severity of "judder".

It has been mentioned in paragraph 3.2 that an error is introduced by basing calculation of friction on the mean velocity v_1 instead of on the variable v_2. The maximum instantaneous value of velocity v_2 can be calculated in the usual way by differentiating twice equation (8), equating to zero, solving for t and inserting this value into the first differential. Dividing the maximum instantaneous velocity v_{2max}, thus obtained, by v_1 a simple velocity ratio results, thus

$$\rho = \frac{v_{2max}}{v_1} = 1 + \sqrt{1 + \left(\frac{R-r}{mv_1\sqrt{\frac{k}{m}}}\right)^2} \qquad (16)$$

It is found in most practical applications that the square of the bracket under the root sign is quite small and the value of ρ tends towards 2 and so the reasoning of paragraph 3.2 (as regards the average value of friction) applies.

It may be of interest to note that due to the presence of friction the inflection point of $x_2 = f(t)$ does not coincide with the intersection with $x_1 = v_1 t$ as one might conclude from the fact that at that instant $x_2 = x_1$ and the spring force equals zero. The inflection point and consequently v_{2max} can be found also graphically from the plot of $x_2 = f(t)$ as the intersection with a line parallel to $x_1 = v_1 t$ (see sketch on page 9)

$$x_{1_F} = v_1 t - \frac{r}{k} \qquad (17)$$

At that instant (of intersection with (17)) the residual spring force just equals the instantaneous value of friction, $\frac{d^2x_2}{dt^2} = 0$ and v_2 is maximum.

In a recent test on a hydraulic servo unit the traces of the output show a remarkable similarity to the curves computed by the present method and the relevant graphs are reproduced here for interest (figure 6).

3. Conclusions

In systems in which power is transmitted from a constant rate source into the output element of finite mass through an elastic medium and in which the output element is restrained by friction, the resulting motion of the mass may be of an incremental character, with periods of rest interposed between the consecutive cycles of sustained amplitude. This period of rest and the relative violence of the incremental motion is strongly accentuated by the difference between the static and kinetic values of friction, which appears to be very pronounced in the case of rubber-metal rubbing combination, such as is used in hydraulic jack seals. Recent investigations have shown that the ratios of the two frictions of the order of 5 to 10 may be frequently realised. The resultant response to a step input is then distinctly incremental, quasi-oscillatory, consisting of a succession of jerks in the general direction of motion and it is submitted that this is the basis of what is commonly referred to as "judder".

There seem to be two main means to mitigate the severity of the phenomenon, namely, one, increasing the own frequency of the system by stiffening the transmitting medium. To this end the fluid should be thoroughly de-aerated by bleeding the system, the jack cylinder so designed and dimensioned as to ensure its maximum rigidity and the pipe-lines of sturdy design be kept as short as practicable. It may be advisable in some systems using stationary piston and moving jack cylinder to reverse the order and keep the cylinder stationary in order to make the use of rigid metal pipelines (in place of elastic flexible hoses) possible. Alternatively, or perhaps simultaneously, the jack friction should be kept generally low and in particular combinations of rubbing materials and lubricants (fluids) with pronounced difference between the static and kinetic friction coefficients should be avoided.

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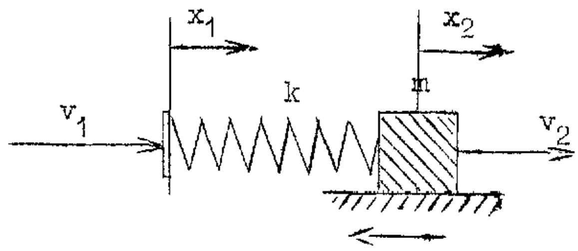
2	F.W. Read	Tests on "Servodyne" Hydraulic flying control assister. Technical Note Mech.Eng.10 (R.A.E. 1947) ARC 12,653
3	H.S. Carslaw T.C. Jaeger	Operational methods in applied mathematics. (Oxford University Press)

APPENDIX I

Equation of motion and derivation of formulae



Phase I

$$x_1 = v_1 t \tag{1}$$

$$v_1 \; t_o \; k = R \tag{2}$$

$$x_{1_o} = v_1 \; t_o = \frac{R}{k} \tag{3}$$

$$t_o = \frac{R}{kv_1} \tag{4}$$

Phase II

Condition of equilibrium of forces acting on the mass m

$$- m \frac{d^2x_2}{dt^2} + F_s - r = 0 \tag{5}$$

Spring force

$$F_s = k \; \Delta x = k \; (x_1 - x_2)$$

If t is now counted from the instant t_o

$$x_1 = x_{1_o} + v_1 t = \frac{R}{k} + v_1 t \tag{6}$$

Thus

$$F_s = k \; (\frac{R}{k} + v_1 t - x_2) \tag{7}$$

Introducing (7) into (5) and rearranging in the way suitable for operational solution

$$\phi (D) = \psi (t)$$

the equation of motion becomes now

$$\frac{d^2x_2}{dt^2} + \frac{k}{m} x_2 = \frac{1}{m} (k\ v_1 t + R-r) \tag{8}$$

Laplace transformation of (8)

$$(p^2 + \frac{k}{m}) \int_0^\infty \ell^{-pt}\ x_2\ dt = \frac{1}{m} \int_0^\infty (k v_1 t + R-r)\ \ell^{-pt}\ dt$$

$$= \frac{1}{m} \left[k\ v_1 \int_0^\infty \ell^{-pt}\ tdt + (R-r) \int_0^\infty \ell^{-pt}\ dt \right] \tag{9}$$

Integrating the right hand side of (9)

$$\int_0^\infty \ell^{-pt}\ tdt = -\frac{1}{p}\ t\ \ell^{-pt} + \frac{1}{p} \int_0^\infty \ell^{-pt}$$

$$= \left[-\frac{1}{p}\ t\ \ell^{-pt} - \frac{1}{p^2}\ \ell^{-pt} \right]_0^\infty$$

$$= \frac{1}{p^2} \tag{10}$$

and

$$\int_0^\infty \ell^{-pt}\ dt = \left[-\frac{1}{p}\ \ell^{-pt} \right]_0^\infty = \frac{1}{p} \tag{11}$$

Thus, inserting (10) and (11) into (9)

$$\int_0^\infty \ell^{-pt}\ x_2\ dt = \frac{kv_1}{m}\ \frac{1}{p^2 (p^2 + \frac{k}{m})} + \frac{R-r}{m}\ \frac{1}{p\ (p^2 + \frac{k}{m})}$$

or

$$\overline{x}_2(p) = v_1 \frac{\frac{k}{m}}{p^2 (p^2 + \frac{k}{m})} + \frac{R-r}{k} \frac{\frac{k}{m}}{p\ (p^2 + \frac{k}{m})} \tag{12}$$

The expressions on the right hand side are reduced by splitting into partial fractions

$$\frac{\frac{k}{m}}{p^2(p^2+\frac{k}{m})} = \frac{k}{m}\left[\frac{A}{p^2} + \frac{B}{p^2+\frac{k}{m}}\right]$$

$$= \frac{k}{m}\;\frac{(A+B)p^2 + A\frac{k}{m}}{p^2(p^2+\frac{k}{m})}$$

$$= \frac{1}{p^2} - \frac{1}{p^2+\frac{k}{m}} \tag{13}$$

Similarly,

$$\frac{\frac{k}{m}}{p(p^2+\frac{k}{m})} = p\;\frac{\frac{k}{m}}{p^2(p^2+\frac{k}{m})} = \frac{1}{p} - \frac{p}{p^2+\frac{k}{m}} \tag{14}$$

Therefore

$$\overline{x}_2(p) = v_1\left[\frac{1}{p^2} - \frac{1}{p^2+\frac{k}{m}}\right] + \frac{R-r}{k}\left[\frac{1}{p} - \frac{p}{p^2+\frac{k}{m}}\right] \tag{15}$$

Using a table of Laplace transforms (Reference 2) the complete solution can now be written directly

$$x_2 = v_1\left[t - \frac{1}{\sqrt{\frac{k}{m}}}\sin\sqrt{\frac{k}{m}}\,t\right] + \frac{R-r}{k}\left[1 - \cos\sqrt{\frac{k}{m}}\,t\right] \tag{16}$$

To determine the maximum amplitude x_2' and the corresponding time t' (measured from the instant t_0) (16) is differentiated and equated to zero

$$\frac{dx_2}{dt} = v_1\left[1 - \cos\sqrt{\frac{k}{m}}\,t\right] + \frac{R-r}{k}\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}\,t = 0 \tag{17}$$

or

$$v_1\cos\sqrt{\frac{k}{m}}\,t = \frac{R-r}{k}\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}\,t + v_1 \tag{18}$$

Squaring (18) and expressing cos in terms of sin

$$v_1^2 - v_1^2 \sin^2\sqrt{\frac{k}{m}}\, t = \left(\frac{R-r}{k}\sqrt{\frac{k}{m}}\right)^2 \sin^2\sqrt{\frac{k}{m}}\, t + 2v_1 \frac{R-r}{k}\sqrt{\frac{k}{m}} \sin\sqrt{\frac{k}{m}}\, t + v_1^2 \tag{19}$$

Dividing by $v_1^2 \sin\sqrt{\frac{k}{m}}\, t$ and rearranging (19)

$$\left[\left(\frac{R-r}{kv_1}\sqrt{\frac{k}{m}}\right)^2 + 1\right] \sin\sqrt{\frac{k}{m}}\, t + 2\,\frac{R-r}{kv_1}\sqrt{\frac{k}{m}} = 0 \tag{20}$$

and

$$\sin\sqrt{\frac{k}{m}}\, t = -\frac{2\,\frac{R-r}{kv_1}\sqrt{\frac{k}{m}}}{\left(\frac{R-r}{kv_1}\sqrt{\frac{k}{m}}\right)^2 + 1} =$$

$$= -\frac{2\,(R-r)\; v_1\; m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1^2\; km} \tag{21}$$

The negative sign signifies that the angle $\sqrt{\frac{k}{m}}\, t$ lies either in the third or fourth quadrant and consequently

$$t' = \frac{2\pi}{\sqrt{\frac{k}{m}}} - \frac{1}{\sqrt{\frac{k}{m}}} \sin^{-1} \frac{2(R-r)\; v_1\, m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1^2\; km} \tag{22}$$

or

$$t' = \frac{\pi}{\sqrt{\frac{k}{m}}} + \frac{1}{\sqrt{\frac{k}{m}}} \sin^{-1} \frac{2(R-r)\; v_1\, m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1^2\; km} \tag{23}$$

To decide between (22) and (23), equation (18) is solved for $\cos\sqrt{\frac{k}{m}}\, t$, thus

$$v_1 - v_1 \cos\sqrt{\frac{k}{m}}\, t + \sqrt{\frac{k}{m}}\; \frac{R-r}{k} \sqrt{1 - \cos^2\sqrt{\frac{k}{m}}\, t} = 0 \tag{24}$$

Writing a instead of $\left(\sqrt{\frac{k}{m}}\ \frac{R-r}{k}\right)$, squaring and rearranging with respect to $\cos\sqrt{\frac{k}{m}}\,t$

$$(v_1^2 + a^2)\cos^2\sqrt{\frac{k}{m}}\,t - 2v_1\cos\sqrt{\frac{k}{m}}\,t + (v_1^2 - a^2) = 0 \qquad (25)$$

a quadratic for $\cos\sqrt{\frac{k}{m}}\,t$ is obtained. Solving for $\cos\sqrt{\frac{k}{m}}\,t$:-

$$\cos\sqrt{\frac{k}{m}}\,t = \frac{2v_1^2 \pm \sqrt{4v_1^4 - 4(v_1^4 - a^4)}}{2(v_1^2 + a^2)}$$

$$= \frac{v_1^2 \pm a^2}{v_1^2 + a^2} \qquad (26)$$

The plus sign must be neglected since $\sqrt{\frac{k}{m}}\,t$ cannot be equal to zero. Taking the minus sign it is clear that $\sqrt{\frac{k}{m}}\,t$ lies in the third quadrant if $v_1 < a$, and in fourth quadrant if $v_1 > a$ and equations (23) or (22) respectively apply.

Introducing t' into (16) the maximum amplitude x_2' is obtained

$$x_2' = v_1\left[t' - \frac{1}{\sqrt{\frac{k}{m}}}\sin\sqrt{\frac{k}{m}}\,t'\right] + \frac{R-r}{k}\left[1 - \cos\sqrt{\frac{k}{m}}\,t'\right] \qquad (27)$$

at that instant

$$x_1' = x_{1_0} + v_1\,t' = \frac{R}{k} + v_1\,t' \qquad (28)$$

The period of the third phase t'' (counted from the instant t') is calculated from the expression

$$t'' = \frac{x_2' - x_1'}{v_1} \qquad (29)$$

x_2' and x_1' are found in equation (27) and (28) respectively.

$$t'' = t' - \frac{1}{\sqrt{\frac{k}{m}}} \sin\sqrt{\frac{k}{m}}\, t' + \frac{R-r}{kv_1}\left[1 - \cos\sqrt{\frac{k}{m}}\, t'\right] - \frac{R}{kv_1} - t'$$

$$= - \frac{1}{\sqrt{\frac{k}{m}}} \sin\sqrt{\frac{k}{m}}\, t' - \frac{R-r}{kv_1} \cos\sqrt{\frac{k}{m}}\, t' - \frac{r}{kv_1} \tag{30}$$

Equations (22) and (23) will supply t' in (30). Let also Ω be written, for the time being, instead of $\sin^{-1} \dfrac{2(R-r)\ v_1\ m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1{}^2 mk}$.

Turning first to equation (22)

$$t'' = - \frac{1}{\sqrt{\frac{k}{m}}} \sin\sqrt{\frac{k}{m}}\left[\frac{2\pi}{\sqrt{\frac{k}{m}}} - \frac{1}{\sqrt{\frac{k}{m}}}\Omega\right] - \frac{R-r}{kv_1} \cos\sqrt{\frac{k}{m}}\left[\frac{2\pi}{\sqrt{\frac{k}{m}}} - \frac{1}{\sqrt{\frac{k}{m}}}\Omega\right] - \frac{r}{kv_1}$$

$$= \frac{1}{\sqrt{\frac{k}{m}}} \sin\Omega - \frac{R-r}{kv_1} \cos\Omega - \frac{r}{kv_1}$$

$$= \frac{1}{\sqrt{\frac{k}{m}}} \sin\Omega - \frac{R-r}{kv_1} \sqrt{1 - \sin^2\Omega} - \frac{r}{kv_1} \tag{31}$$

Taking next equation (23)

$$t'' = - \frac{1}{\sqrt{\frac{k}{m}}} \sin\sqrt{\frac{k}{m}}\left[\frac{\pi}{\sqrt{\frac{k}{m}}} + \frac{1}{\sqrt{\frac{k}{m}}}\Omega\right] - \frac{R-r}{kv_1} \cos\sqrt{\frac{k}{m}}\left[\frac{\pi}{\sqrt{\frac{k}{m}}} + \frac{1}{\sqrt{\frac{k}{m}}}\Omega\right] - \frac{r}{kv_1}$$

$$= \frac{1}{\sqrt{\frac{k}{m}}} \sin\Omega + \frac{R-r}{kv_1} \cos\Omega - \frac{r}{kv_1}$$

$$= \frac{1}{\sqrt{\frac{k}{m}}} \sin\Omega + \frac{R-r}{kv_1} \sqrt{1 - \sin^2\Omega} - \frac{r}{kv_1} \tag{32}$$

Remembering that

$$\sin(\sin^{-1} A) = A$$

$$t'' = \frac{2(R-r)\ v_1 m}{(R-r)^2 + v_1^2\ m\,k} \mp \frac{R-r}{kv_1}\sqrt{1 - \left(\frac{2(R-r)\ v_1\ m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1^2\ m\,k}\right)^2} - \frac{r}{kv_1}$$

$$= \frac{2(R-r)\ v_1 m}{(R-r)^2 + v_1^2\ m\,k} \mp \frac{R-r}{kv_1}\sqrt{\frac{\left\{\mp\left[(R-r)^2 - v_1^2\ m\,k\right]\right\}^2}{\left[(R-r)^2 + v_1^2\ m\,k\right]^2}} - \frac{r}{kv_1} \tag{33}$$

Taking first the top signs

$$t'' = \frac{2(R-r)\ v_1 m}{(R-r)^2 + v_1^2\ m\,k} - \frac{1}{kv_1}\left\{(R-r)\ \frac{-(R-r)^2 + v_1^2\ m\,k}{(R-r)^2 + v_1^2\ m\,k} + r\right\}$$

$$= \frac{2(R-r)\ v_1 m}{(R-r)^2 + v_1^2\ m\,k} - \frac{1}{kv_1}\ \frac{(R-r)^2(2r-R) + Rv_1^2\ m\,k}{(R-r)^2 + v_1^2\ m\,k}$$

$$= \frac{2(R-r)\ v_1^2 m\ k - (R-r)^2(2r-R) + Rv_1^2 m\,k}{kv_1\left[(R-r)^2 + v_1^2\ m\,k\right]}$$

$$= \frac{(R-2r)\ v_1^2\ m\,k + (R-2r)(R-r)^2}{kv_1\ \left[(R-r)^2 + v_1^2\ m\,k\right]}$$

$$= \frac{R-2r}{kv_1} \tag{34}$$

The same result is obtained if the bottom signs are taken in equation (33).

It is obvious that t'' becomes zero when $R = 2r$. The total period of the complete cycle is

$$T = t_o + t' + t'' \tag{35}$$

Referring to equations (4), (22) or (23) and (34)

$$T = \frac{R}{kv_1} + \frac{2\pi}{\sqrt{\frac{k}{m}}} - \frac{1}{\sqrt{\frac{k}{m}}} \sin^{-1} \frac{2(R-r)\ v_1\ m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1^2\ k\,m} + \frac{R-2r}{kv_1}$$

$$= \frac{2\pi}{\sqrt{\frac{k}{m}}} + \frac{2(R-r)}{kv_1} - \frac{1}{\sqrt{\frac{k}{m}}} \sin^{-1} \frac{2(R-r)\ v_1\ m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1^2\ k\,m} \qquad (36)$$

or

$$T = \frac{R}{kv_1} + \frac{\pi}{\sqrt{\frac{k}{m}}} + \frac{1}{\sqrt{\frac{k}{m}}} \sin^{-1} \frac{2(R-r)\ v_1\ m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1^2\ k\,m} + \frac{R-2r}{kv_1}$$

$$= \frac{\pi}{\sqrt{\frac{k}{m}}} + \frac{2(R-r)}{kv_1} + \frac{1}{\sqrt{\frac{k}{m}}} \sin^{-1} \frac{2(R-r)\ v_1\ m\sqrt{\frac{k}{m}}}{(R-r)^2 + v_1^2\ k\,m} \qquad (37)$$

The velocity of the mass m at any instant is given by the first derivative of (16) and appears in equation (17)

$$\frac{dx_2}{dt} = v_1 \left[1 - \cos\sqrt{\frac{k}{m}}\,t\right] + \frac{R-r}{k}\sqrt{\frac{k}{m}} \sin\sqrt{\frac{k}{m}}\,t$$

Differentiating again and equating to zero

$$\frac{d^2x_2}{dt^2} = v_1\sqrt{\frac{k}{m}} \sin\sqrt{\frac{k}{m}}\,t + \frac{R-r}{k}\,\frac{k}{m} \cos\sqrt{\frac{k}{m}}\,t = 0 \qquad (38)$$

$$\frac{R-r}{m} \cos\sqrt{\frac{k}{m}}\,t = - v_1\sqrt{\frac{k}{m}} \sin\sqrt{\frac{k}{m}}\,t$$

or

$$\tan\sqrt{\frac{k}{m}}\,t = - \frac{R-r}{m\ v_1\sqrt{\frac{k}{m}}} \qquad (39)$$

The instant of maximum instantaneous velocity v_2 is calculated (t counted from the instant t_o) from

$$t_{v_{2max}} = \frac{1}{\sqrt{\frac{k}{m}}} \tan^{-1} \frac{-(R-r)}{m\, v_1 \sqrt{\frac{k}{m}}}$$

$$= \frac{\pi}{\sqrt{\frac{k}{m}}} - \frac{1}{\sqrt{\frac{k}{m}}} \tan^{-1} \frac{R-r}{m\, v_1 \sqrt{\frac{k}{m}}} \tag{40}$$

and maximum velocity

$$v_{2max} = v_1 \left[1 - \cos\sqrt{\frac{k}{m}}\, t_{v_{2max}}\right] + \frac{R-r}{k}\sqrt{\frac{k}{m}} \sin\sqrt{\frac{k}{m}}\, t_{v_{2max}}$$

$$= v_1 \left[1 - \cos\left(\pi - \tan^{-1}\frac{R-r}{m\, v_1\sqrt{\frac{k}{m}}}\right)\right] + \frac{R-r}{m}\sqrt{\frac{k}{m}} \sin\left(\pi - \tan^{-1}\frac{R-r}{m\, v_1\sqrt{\frac{k}{m}}}\right)$$

$$= v_1 \left[1 + \cos \tan^{-1}\frac{R-r}{m\, v_1\sqrt{\frac{k}{m}}}\right] + \frac{R-r}{k}\sqrt{\frac{k}{m}} \sin \tan^{-1}\frac{R-r}{m\, v_1\sqrt{\frac{k}{m}}} \tag{41}$$

Remembering again that

$$\sin(\tan^{-1} A) = \frac{A}{\sqrt{1 + A^2}}$$

and

$$\cos(\tan^{-1} A) = \frac{1}{\sqrt{1 + A^2}}$$

$$v_{2max} = v_1 \left[1 + \frac{1}{\sqrt{1 + \left(\frac{R-r}{m\, v_1\sqrt{\frac{k}{m}}}\right)^2}}\right] + \frac{R-r}{k}\sqrt{\frac{k}{m}}\, \frac{\frac{R-r}{m\, v_1\sqrt{\frac{k}{m}}}}{\sqrt{1 + \left(\frac{R-r}{m\, v_1\sqrt{\frac{k}{m}}}\right)^2}} \tag{42}$$

and the ratio

$$\rho = \frac{v_{2max}}{v_1} = 1 + \frac{1 + \frac{(R-r)^2}{k\,m\,v_1^2}}{\sqrt{1 + \left(\frac{R-r}{m\,v_1\sqrt{\frac{k}{m}}}\right)^2}}$$

$$= 1 + \sqrt{1 + \left(\frac{R-r}{m\,v_1\sqrt{\frac{k}{m}}}\right)^2} \qquad (43)$$

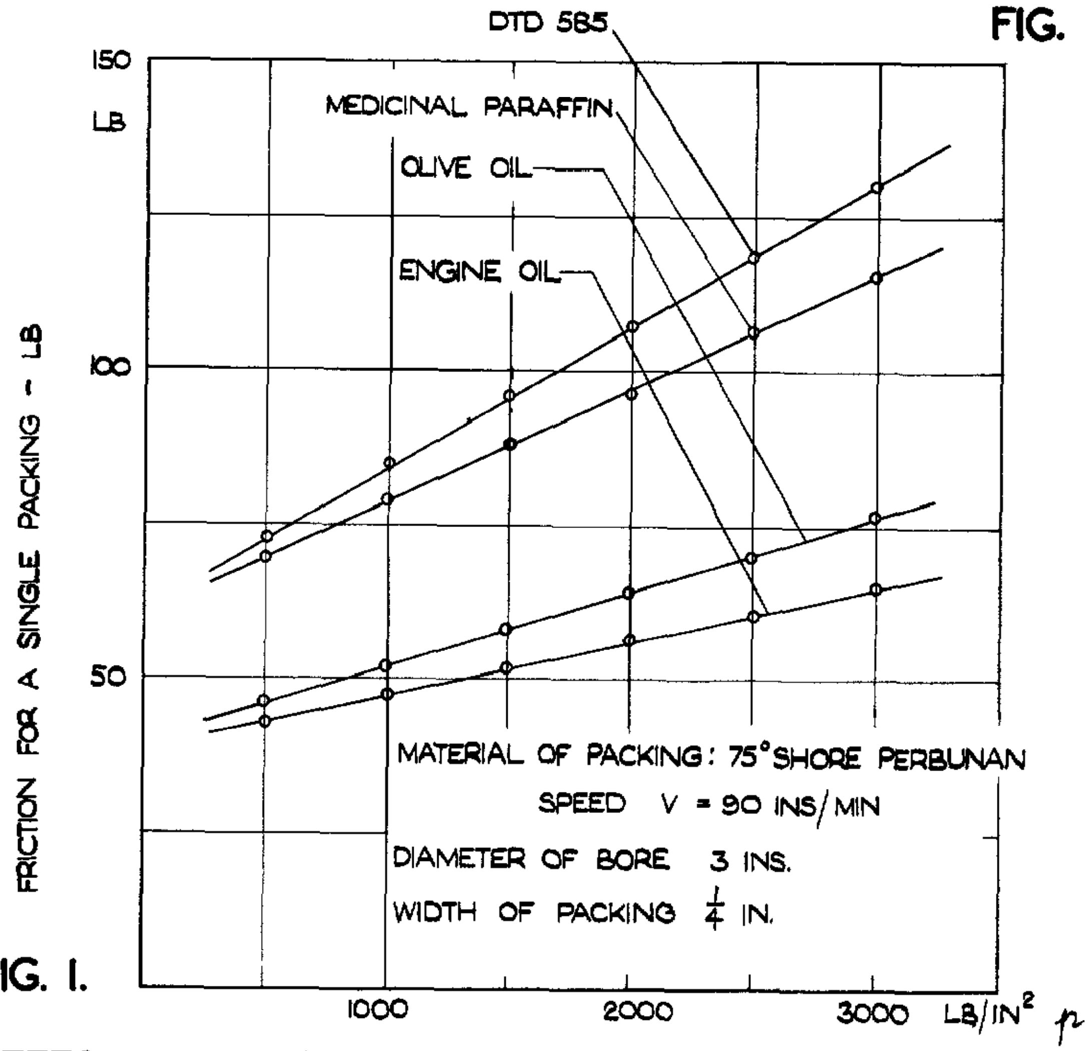


FIG. 1.

EFFECT ON FRICTION OF WORKING FLUID & PRESSURE.

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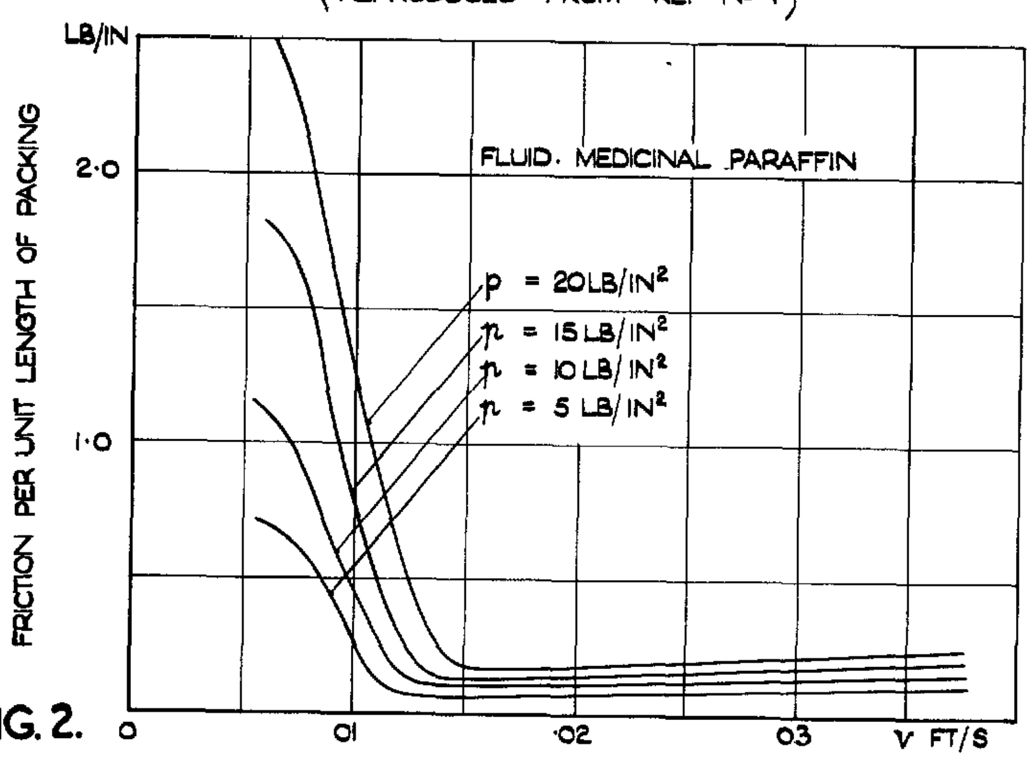

FIG. 2.

FRICTION VALUES AT THE TRANSITION BETWEEN BOUNDARY & FILM LUBRICATION

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FIG. 3

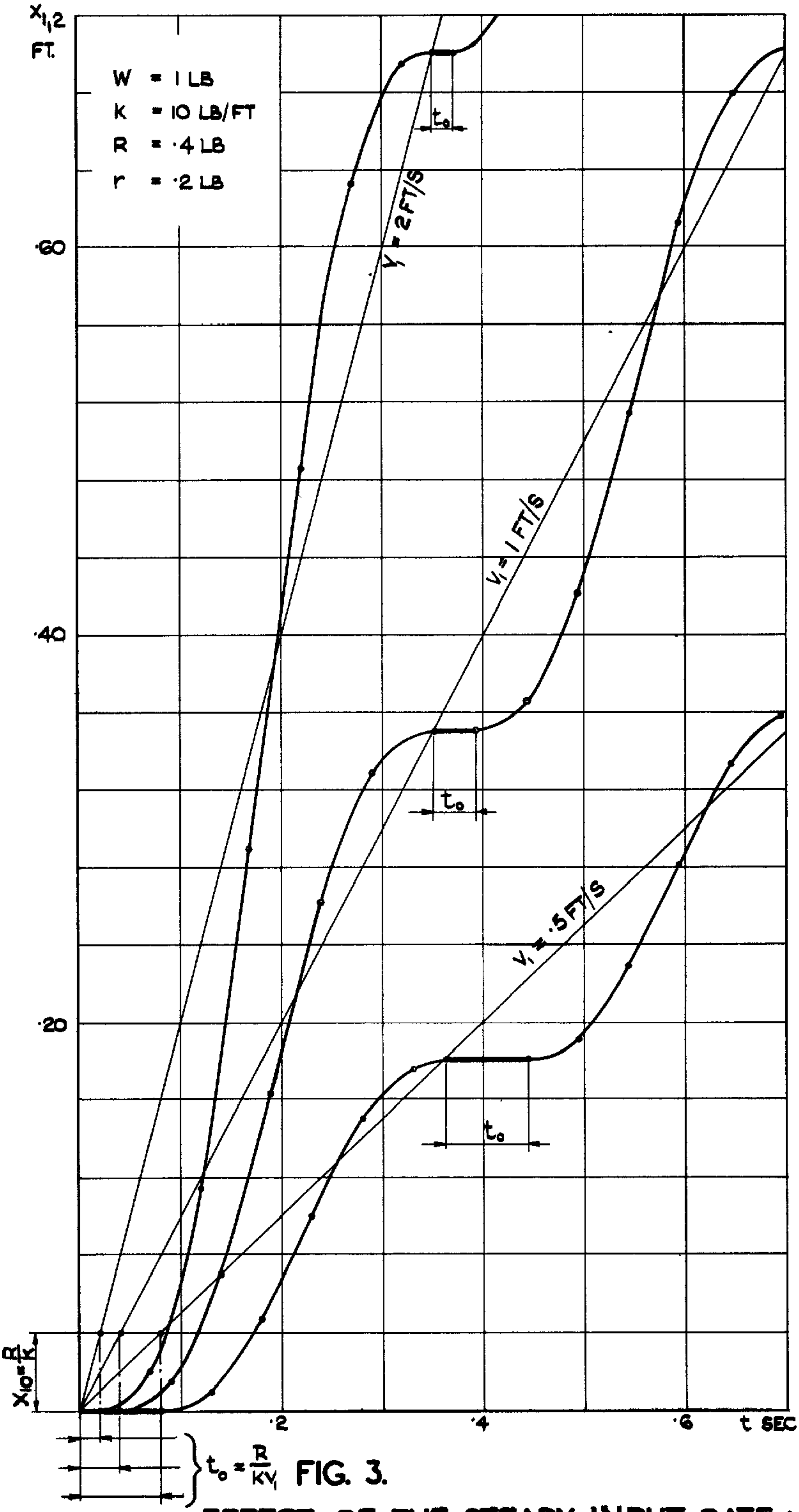

FIG. 3.

EFFECT OF THE STEADY INPUT RATE V_1

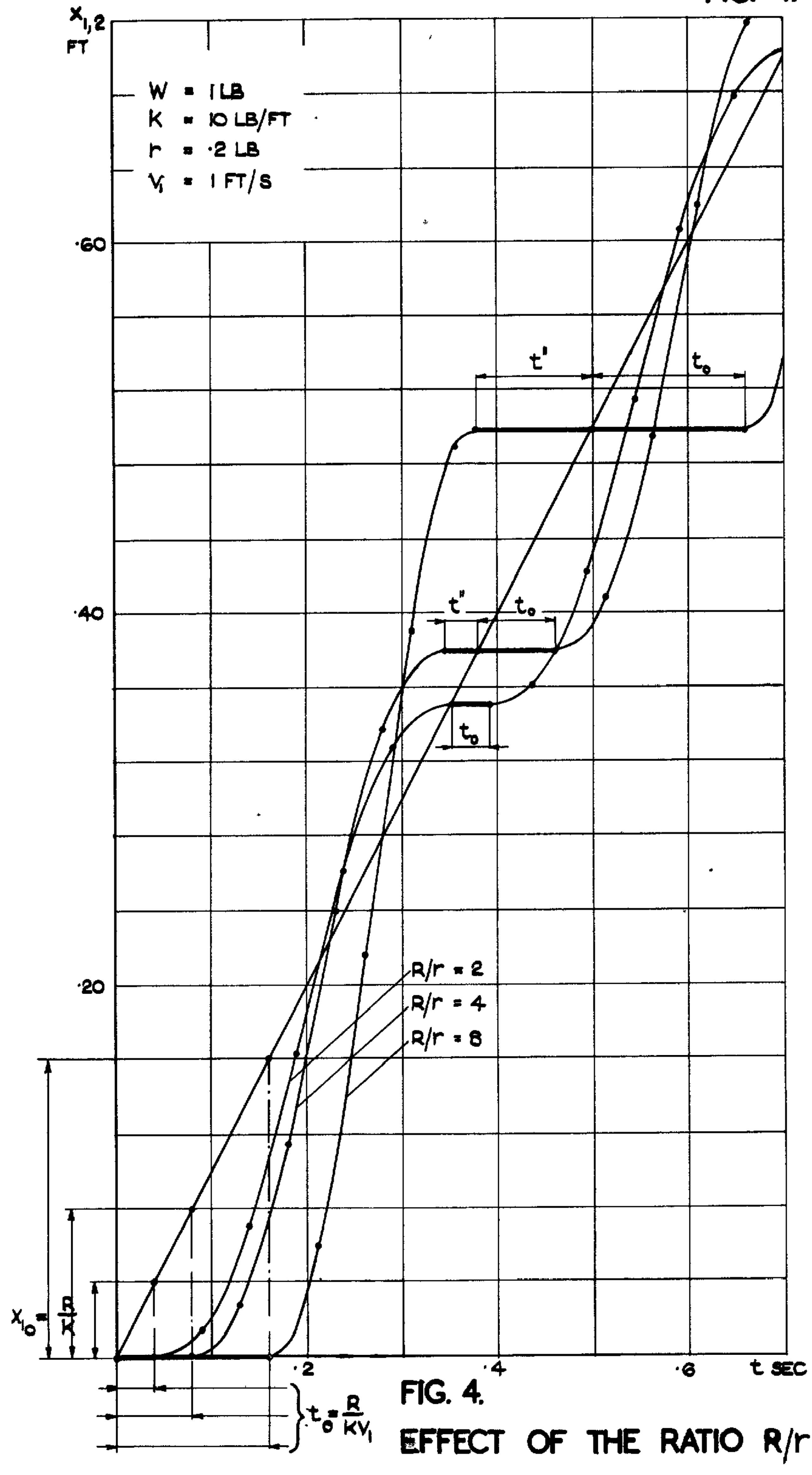

FIG. 4.

EFFECT OF THE RATIO R/r

FIG. 5.

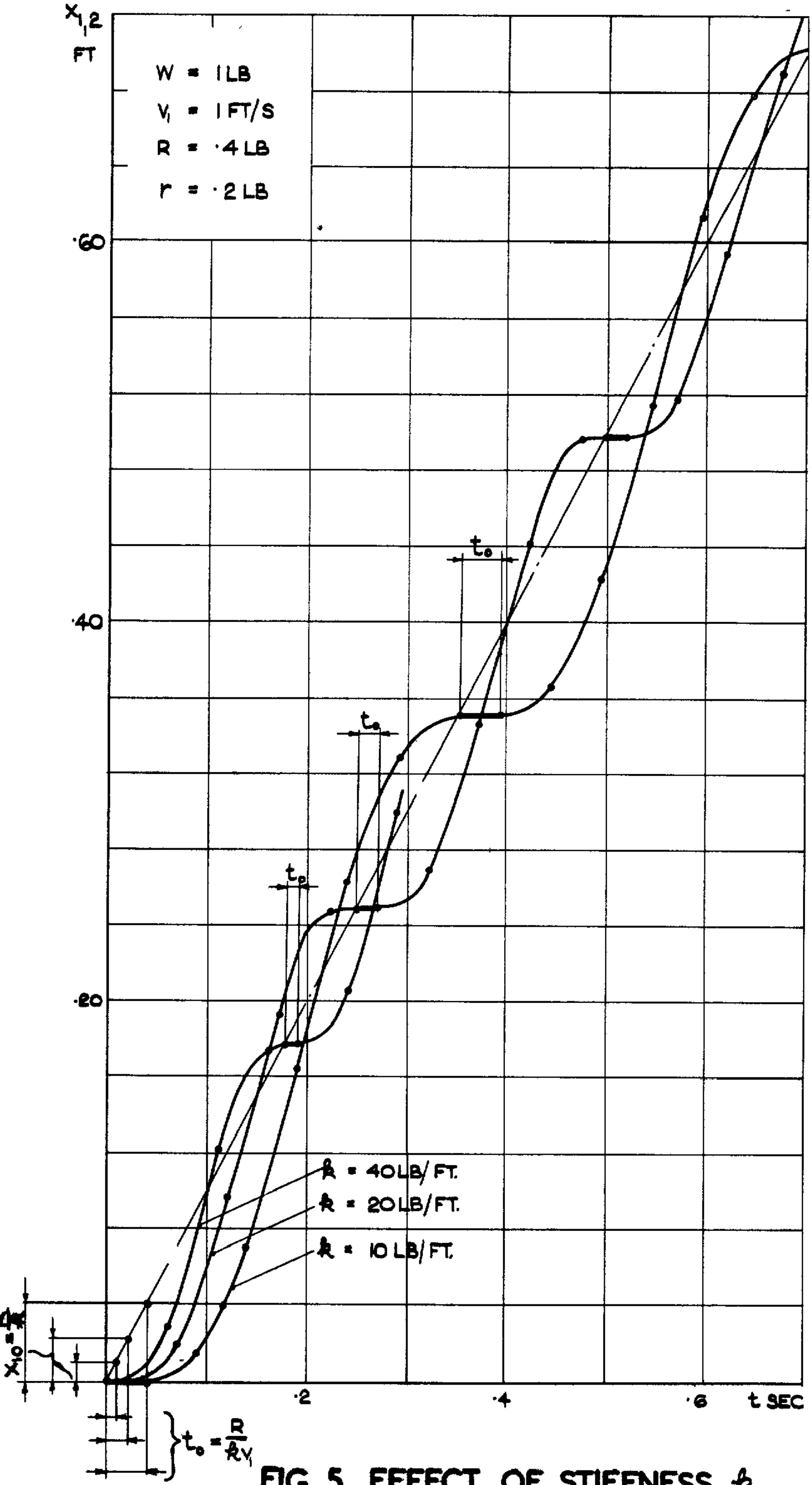

FIG. 5. EFFECT OF STIFFNESS k

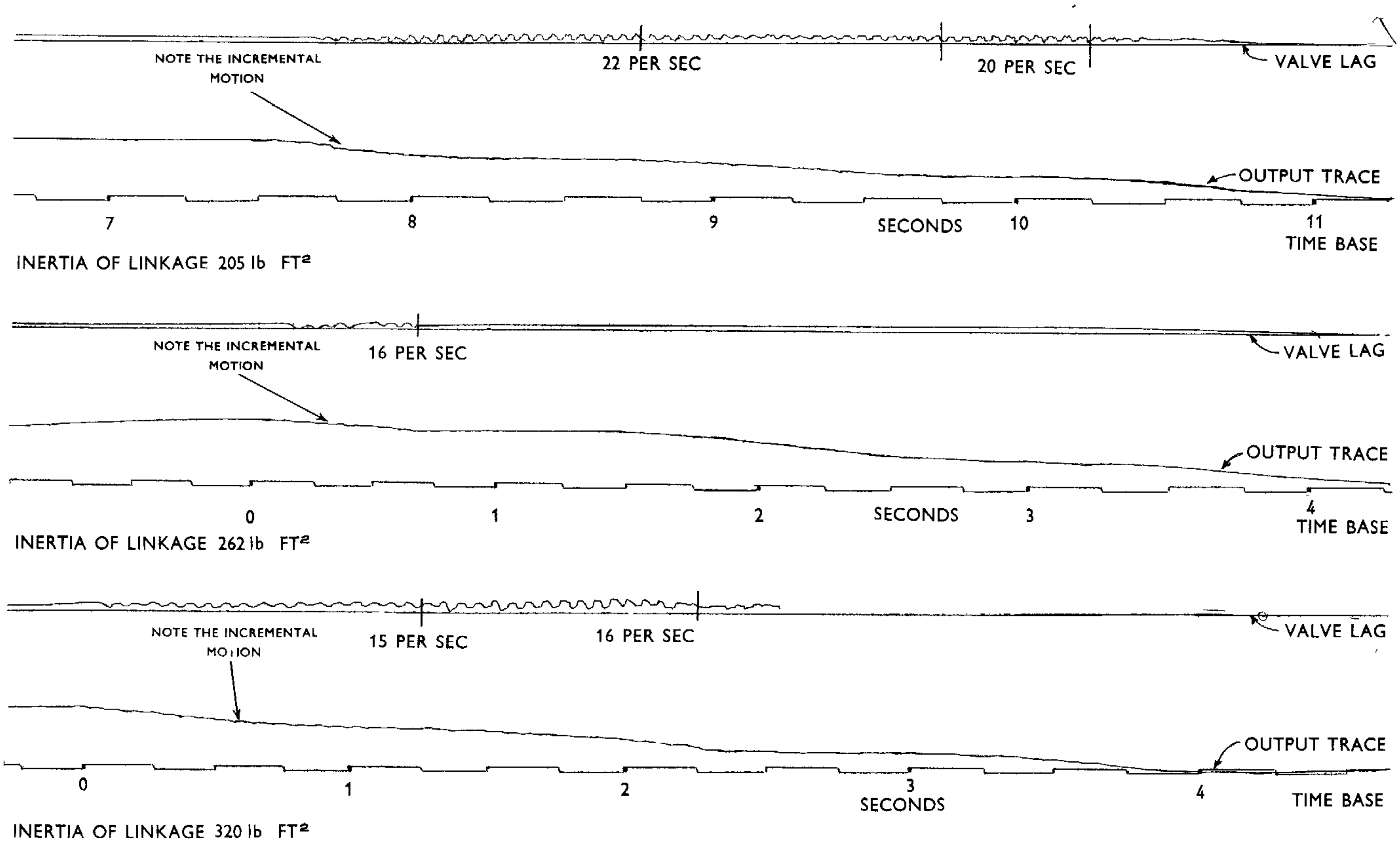

FIG.6. LAG AND OUTPUT TRACE DURING JUDDER ON A FOLLOWING LOAD. DECREASING FROM 500 lb. MAX.

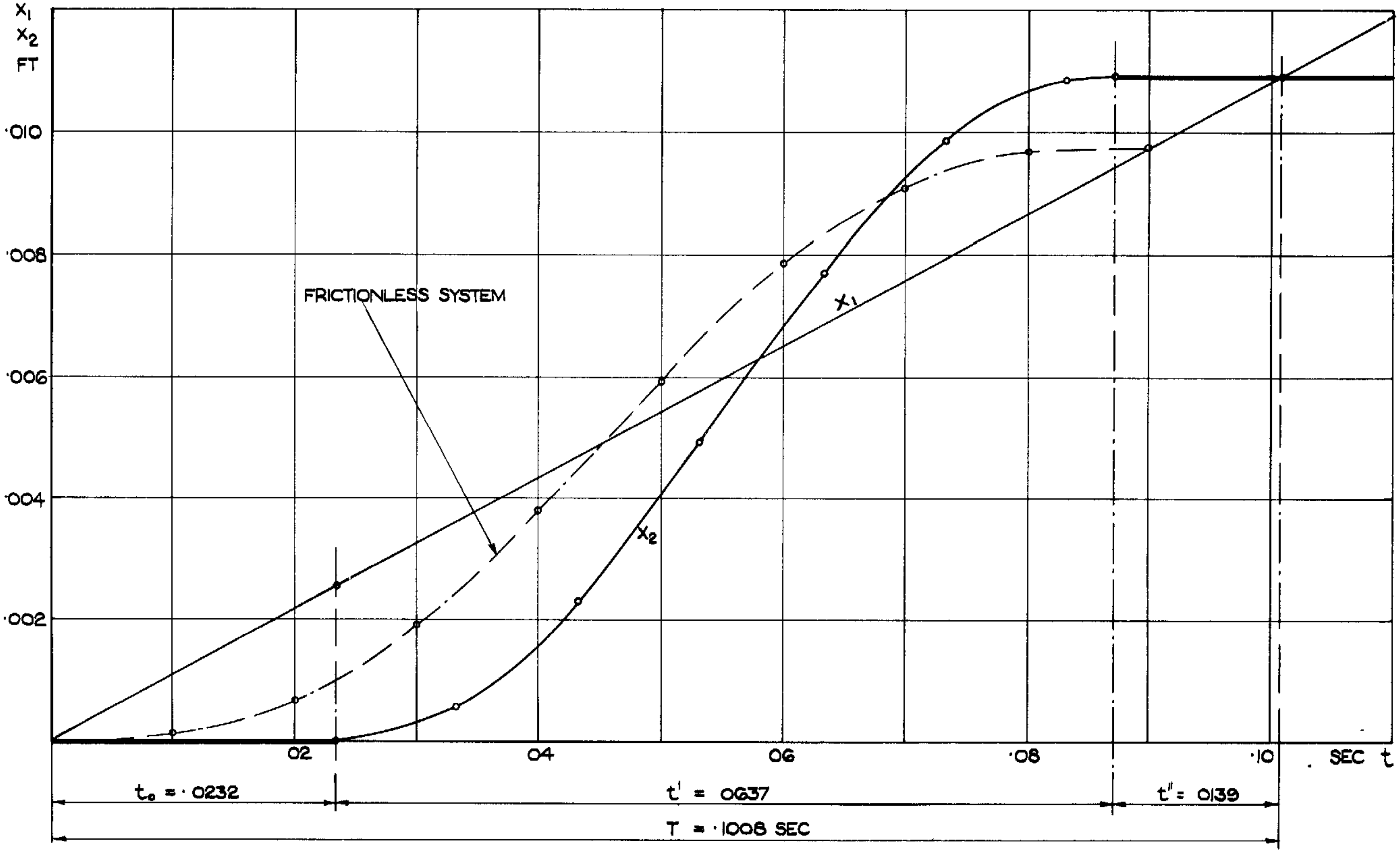

FIG. 7. HYDRAULIC SERVO INSTALLATION.

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