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Stress Considerations in the Design of Pressurised Shells

By

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Stress considerations in the design of pressurised shells

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E. H. Mansfield

SUMMARY

This report considers the design and stress analysis of pressurised thin-walled shells with special reference to openings in the shell wall.

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1 Introduction

In a pressurised shell the pressure is resisted by the tensile or membrane stiffness of the shell wall acting alone or in combination with the flexural stiffness of the shell wall. If the flexural stiffness is not called into play in resisting the pressure the calculation of the membrane stresses depends only on considerations of equilibrium. If the flexural stiffness is called into play an accurate calculation of the stresses is impracticable[1] unless there is axial symmetry, as in the spherical shell with a reinforced circular opening considered in Section 3.

In this report the influence on the membrane and flexural stresses of stringers, frames, end caps and reinforced openings is investigated and the design of end caps and openings which cause no stress concentration is considered.

2 List of Symbols

Structure properties:

E = Young's modulus

ν = Poisson's ratio, assumed to be 0.3

r = radius of sphere or cylinder

t = wall thickness

α = semi-angle subtended by circular hole in sphere

β = semi-angle subtended by cone

A = cross-sectional area of centrally placed reinforcing member

$\bar{A}$ = cross-sectional area of eccentrically placed reinforcing member

A^* = value of A appropriate to a neutral hole

A_F = cross-sectional area of frame

w = frame pitch

b = stringer pitch

I_s = moment of inertia of each stringer-cum-adjacent-skin

h = distance from neutral axis of I_s to inside edge of stringer

a = half the shorter axis of a neutral hole

Z = distance from cone apex to centre of neutral hole

r_1 = depth of end cap

r_n = radius of curvature of reinforcing member normal to shell surface

I_n = is introduced after equation (39)

loads and stresses	p	=	hydrostatic pressure in shell
	p_{cr}	=	value of p to cause buckling of inverted end caps
	σ_ϕ	=	stress in sphere
	σ_θ	=	stress in sphere or hoop stress in cylinders or cones
	σ_x	=	longitudinal stress in cylinder
	σ_z	=	stress along generator in cone
	σ_{ring}	=	stress in reinforcing ring
	σ_s	=	stringer stress
	$\sigma_{x,b}$	=	longitudinal stress due to bending of shell wall
	$\sigma_{s,b}$	=	stringer stress due to longitudinal bending
	M_x	=	bending moment in stringer-cum-adjacent-skin
	H	=	horizontal reaction per unit length between ring and shell
	P	=	load in reinforcing member
	N	=	normal reaction per unit length between window and reinforcing member
	Δ	=	radial displacement in plane of circular ring
	ω	=	rotation of cross-section of reinforcement
axes	Ox	=	longitudinal axis
	Oz	=	axis along generator of cone
	Oy	=	axis normal to cylinder $\mathrm{C_L}$ (Section 4.2), or axis normal to Ox and in the developed surface of cylindrical shell (Section 4.3) or axis normal to plane of shell wall (Section 6)
	dS	=	element of area
non-dimensional parameters	n	=	ratio of thickness of cap to thickness of wall of spherical shell
	S	=	cross-sectional area of stringer + (bt)
	F	=	effective cross-sectional area of frame + (wt)
	μ	=	$\dfrac{1.3w}{\sqrt{(rt)}}$
	η	=	$w\left(\dfrac{bt}{4r^2 I_s}\right)^{\frac{1}{4}}$
	λ	=	$\dfrac{a}{Z}$
	ε	=	efficiency of eccentrically placed reinforcing member
	K	=	(shear stress)/(hoop stress)

3 Spherical Shell

3.1 Spherical shell with reinforced circular opening

Consider two spherical shells with reinforced circular openings. The window in the first shell is flat and that in the second has the same curvature as the shell. In both cases the function of the windows is to resist the pressure by their flexural rigidity and to transmit the resultant pressure by bearing (without friction, say) on to the reinforcing ring.

If the flexural stiffness of the shell wall is neglected the membrane stresses are determined by

$$\left.\begin{aligned}\sigma_\phi &= \sigma_\theta \\ &= \frac{pr}{2t}\end{aligned}\right\} \tag{1}$$

and the direct stress in the reinforcing ring is given by

$$\sigma_{ring} = \frac{p\,r^2 \sin\alpha \cos\alpha}{2A} \tag{2}$$

for the flat window, and

$$\sigma_{ring} = \frac{p\,r^2 \tan\alpha}{2A} \tag{3}$$

for the curved window.

Now the radial displacement in the plane of the ring is given by

$$\Delta_{ring} = \left(\frac{r \sin\alpha}{E}\right) \sigma_{ring} \tag{4}$$

for the ring, and

$$\Delta_{shell} = \left(\frac{r \sin\alpha}{E}\right) (\sigma_\theta - \nu\sigma_\phi) \tag{5}$$

for the adjacent shell, and unless

$$\Delta_{ring} = \Delta_{shell} \tag{6}$$

there will be an additional interaction between ring and shell the magnitude of which will depend on the flexural stiffness of the shell wall. It is shown in Appendix I that this interaction between ring and shell is given by

$$H = \frac{\frac{pr}{2}\left(\frac{\cos\alpha}{A} - \frac{1-\nu}{rt\sin\alpha}\right)}{\frac{1}{A} + 2\,\{3(1-\nu^2)\}^{\frac{1}{4}}\, r^{-\frac{1}{2}}\, t^{-3/2}} \tag{7}$$

for the flat window, and

$$H = \frac{\frac{pr}{2}\left(\frac{\sec\alpha}{A} - \frac{1-\nu}{rt\sin\alpha}\right)}{\frac{1}{A} + 2\,\{3(1-\nu^2)\}^{\frac{1}{4}}\, r^{-\frac{1}{2}}\, t^{-3/2}} \tag{8}$$

for the curved window.

If

$$\left.\begin{aligned} \alpha &< 15^\circ \\ \frac{r}{t} &> 500 \end{aligned}\right\} \tag{9}$$

equations (7) and (8) approximate to

$$H \approx 0.2p\,(rt)^{3/2}\left\{\frac{1}{A} - \frac{1-\nu}{rt\sin\alpha}\right\} . \tag{10}$$

Note that if H is zero,

$$\left.\begin{aligned} A &= A^* \\ &= \frac{rt\sin\alpha}{1-\nu} \end{aligned}\right\} \tag{11}$$

which is the amount of reinforcement necessary to make the hole neutral[2] in an equivalent flat sheet under "hydrostatic" tension.

The interaction H causes bending stresses in the shell and a stress concentration factor which may be put in the form

$$\text{s.c.f.} \approx 1 + 0.4\left|\frac{A^*}{A} - 1\right| . \tag{12}$$

3.2 Spherical shell with cap of greater thickness

The presence of a cap of greater thickness on a spherical shell will reduce the membrane stresses in the cap, but it will also cause stresses due to bending along the common boundary of cap and sphere. It is shown in Appendix II that due to this bending of the shell wall there is a stress concentration factor for the thinner sheet equal to

$$1 + 1.3\left[\frac{n\,(n+1)(n-1)^2}{1 + 2\,(n^{3/2} + n^2 + n^{5/2}) + n^4}\right] . \tag{13}$$

For example, if the cap has twice the thickness of the shell there is a stress concentration factor in the shell equal to 1.2. If the shell is built-in the stress concentration factor, obtained by letting $n \to \infty$ in equation (13), is 2.3. Note that equation (13) is independent of the extent of the cap.

4 Cylindrical shells

In considering cylindrical shells it is convenient to consider first the stress distribution in the main body of the shell and the influence of stringer and frame reinforcement. The effect of the type of end caps for the cylinder will be considered next and finally the design of openings in the wall of the cylinder.

4.1 Stresses in the main body of the shell

4.11 Unreinforced shell

The hoop and longitudinal stresses are given by

$$\left.\begin{aligned} \sigma_\theta &= \frac{pr}{t} \\ \sigma_x &= \frac{pr}{2t} \end{aligned}\right\} \qquad (14)$$

4.12 Shell reinforced by stringers and closely spaced frames

Because of the effect of Poisson's ratio the greatest stresses[3] occur in the shell rather than in the stringers or frames:

$$\sigma_\theta = \frac{pr}{t}\left\{\frac{1 + S + 0.15\,F}{1 + S + F + 0.9\,SF}\right\} \qquad (15)$$

and

$$\sigma_x = \frac{pr}{2t}\left\{\frac{1 + F + 0.6\,S}{1 + S + F + 0.9\,SF}\right\} . \qquad (16)$$

The stringer stress is given by

$$\sigma_s = \frac{pr}{2t}\left\{\frac{0.4 + 0.9\,F}{1 + S + F + 0.9\,SF}\right\} . \qquad (17)$$

It will be seen that the stringers are inefficient in reducing the longitudinal stress in the shell. For example, if

$$S = 0.5$$

and

$$F = 0$$

it follows from equation (16) that

$$\sigma_x = 0.87 \left(\frac{pr}{2t}\right)$$

instead of $0.67 \left(\frac{pr}{2t}\right)$ if the stringers were fully effective.

The frames are apparently more efficient in reducing the hoop stress. For example, if

$$F = 0.5$$

and

$$S = 0$$

it follows from equation (15) that

$$\sigma_\theta = 0.72 \frac{pr}{t} .$$

However, in equations (15), (16) and (17) the symbol F refers to the effective cross-sectional area of the frames in resisting hoop stress. For Z-section and similar frame sections the hoop stress will vary considerably across the section of the frame because of the bending flexibility of the frame in the plane of its cross-section; outstanding legs of a frame may be almost unstressed. The presence of stringer cut-outs in the frames will also reduce their effectiveness. An average value for F may be given by

$$F = \frac{0.5 A_F}{wt} \qquad (18)$$

and it will then be seen that the frames are as inefficient in reducing the hoop stress as stringers are in reducing the longitudinal stress.

4.13 <u>Shell reinforced by widely spaced frames (no stringers)</u>

If the frames are widely spaced their action in reducing the hoop stress in the shell will be localised to regions near the frames. In addition localised longitudinal bending of the shell wall will occur and the peak stresses so developed may exceed the hoop stress in the shell. It is shown in Appendix III that the peak stress due to this longitudinal bending is given by

$$\sigma_{x,b} = \frac{2pr}{t} \left(\frac{\sinh \mu - \sin \mu}{\sinh \mu + \sin \mu}\right) \left\{ \frac{1}{1 + \frac{1.5 \sqrt{rt}}{Fw} \left(\frac{\cosh \mu - \cos \mu}{\sinh \mu + \sin \mu}\right)} \right\} \qquad (19)$$

where

$$\mu = \frac{1.3w}{\sqrt{rt}} .$$

For most shells, μ will exceed 5, and equation (19) simplifies to:

$$\sigma_{x,b} = \frac{2pr}{t\left(1 + \frac{1.5\sqrt{rt}}{Fw}\right)} \quad . \qquad (20)$$

For example, if

$$\frac{w}{r} = 0.3$$

$$F = 0.1$$

$$\frac{r}{t} = 2500$$

it will be found that

$$\sigma_{x,b} = \frac{pr}{t}$$

so that

$$\sigma_x + \sigma_{x,b} = 1.5\,\frac{pr}{t}$$

which represents a stress concentration factor of 1.5. For a corresponding cylinder with F equal to 0.2 the stress concentration factor is 1.83. For a completely rigid frame the stress concentration factor is 2.5.

4.14 Shell reinforced by widely spaced frames (with stringers)

The presence of stringers stiffens the wall of the shell considerably against the longitudinal bending discussed in Section 4.13.

It is shown in Appendix IV that the longitudinal bending moment acting on each stringer-cum-adjacent-skin has a maximum value given by

$$M_x = 0.85\,pr\left(\frac{b\,I_s}{t}\right)^{\frac{1}{2}}\left(\frac{\sinh\eta - \sin\eta}{\sinh\eta + \sin\eta}\right)\left\{\frac{1}{1 + \frac{2}{F\eta}\left(\frac{\cosh\eta - \cos\eta}{\sinh\eta + \sin\eta}\right)}\right\} \qquad (21)$$

where

$$\eta = w\left(\frac{bt}{4r^2\,I_s}\right)^{\frac{1}{4}} .$$

The maximum tensile stress due to this longitudinal bending, which must be added to the value determined by equation (17), is given by

$$\sigma_{s,b} = \frac{h\,M_x}{I_s} \qquad (22)$$

and it will occur in the stringers at points furthest from the wall of the shell. For the worst possible case, that of a single rigid frame, we find

$$\sigma_{s,b} = 0.85\ phr \left(\frac{b}{t\ I_s}\right)^{\frac{1}{2}} . \qquad (23)$$

4.2 The effect of end caps

At the junction of the cylinder with the end caps there will be a localised bending of the walls of the shell due to the different degrees of hoop expansion due to the membrane stresses in the cylinder and the end caps. The effect of spherical and ellipsoidal end caps is considered by Timoshenko[1] who shows that, for an unreinforced cylinder, the hoop stress in the cylinder walls is the determining factor in design.

4.21 Stress concentrations

If the wall thicknesses of cylinder and end cap are the same, it is found that

$$\text{s.c.f.} = 1.032$$

for a hemispherical cap, and

$$\text{s.c.f.} \doteq 1 + 0.032 \left(\frac{r}{r_1}\right)^2 \qquad (24)$$

for an ellipsoidal cap.

Similarly, it can be shown that

$$\text{s.c.f.} = 1.077$$

for an inverted hemispherical cap, and

$$\text{s.c.f.} = 1 + 0.032 \left| 3.4 - \left(\frac{r}{r_1}\right)^2 \right| \qquad (25)$$

for an inverted ellipsoidal cap.

4.22 Designs without stress concentrations

There will be no localised bending of the walls of the shell if the hoop expansions due to membrane stresses in the cylinder and the end caps are the same. This may be achieved in a number of ways; for example:-

(i) by taking the thickness of the wall of the hemisphere equal to 0.41 times the cylinder wall thickness; but this gives a greater membrane stress in the hemisphere and it would be better to compromise with a ratio of wall thicknesses of about 0.6;

(ii) by taking an inverted ellipsoidal cap in which

$$r_1 = 0.54\ r \quad \text{(see equation (25))};$$

(iii) by designing the cap so that there will be no abrupt change in the curvature of the shell wall. A suitable form for such a cap would be the surface of revolution of the curve

$$\left(\frac{y}{r}\right)^2 + \left(\frac{x}{r_1}\right)^3 = 1 \qquad (26)$$

and the optimum value of r_1 which makes cylinder and cap equally strong is given by

$$r_1 = 0.75\ r.$$

4.23 Buckling of inverted caps

The use of inverted caps may be ruled out because of the possibility of failure precipitated by buckling. It is shown by Timoshenko that the critical buckling pressure of a spherical shell is given by

$$p_{cr} = 1.2\ E \left(\frac{t}{r}\right)^2 \qquad (27)$$

and this formula may be used with fair accuracy for ellipsoidal and other shells if the maximum spherical radius of curvature is substituted for r.

4.3 Design of openings in the wall of the cylinder

It was shown in Section 3.1 that for the spherical shell the shape of the opening and the type of the reinforcement that caused zero stress concentration and zero bending of the walls of the shell corresponded to a neutral hole in plane sheet, provided that the radius of the hole was small in comparison with the radius of the spherical shell. It is shown in Appendix V that for any shell whose walls are developable the type of opening to cause zero stress concentration and zero bending of the walls of the shell corresponds exactly to the neutral hole in the developed shell; there is no limitation on the size of the hole in comparison with the size of the shell.

4.31 The shape of the neutral hole

The neutral hole in the cylindrical shell will have the form in the developed plane of an ellipse with axes in the ratio $\sqrt{2} : 1$, the longer axis lying in the direction of the greater (i.e. hoop) stress. If the length of the shorter axis is $2a$ the equation determining the shape of the opening is

$$\frac{x^2}{a^2} + \frac{y^2}{2a^2} = 1. \qquad (28)$$

4.32 Section area of the reinforcing member

Unless the shell is reinforced by stringers or frames the cross-sectional area of the reinforcement round the opening is given by

$$\frac{A^*}{at} = \frac{\sqrt{2}\left\{1 + \left(\frac{x}{a}\right)^2\right\}^{3/2}}{0.4 + 3\left(\frac{x}{a}\right)^2}. \qquad (29)$$

(The 0.4 comes from $(1-2\nu)$ with ν equal to 0.3.)

If the shell is reinforced by closely spaced stringers and frames

$$\frac{A^*}{at} = \frac{\sqrt{2}\,(1 + S + F + 0.9SF)\left\{1 + \left(\frac{x}{a}\right)^2\right\}^{3/2}}{0.4 + 0.9F + (3 + 3.6S - 0.9F)\left(\frac{x}{a}\right)^2} \;. \tag{30}$$

In practice it will be sufficiently accurate to have a constant reinforcement of magnitude $(A^*)_{x=a}$.

4.33 The load in the reinforcing member

The load in the reinforcing member is given by

$$P = p\,r\,a\left\{\frac{1 + \left(\frac{x}{a}\right)^2}{2}\right\}^{\frac{1}{2}} \;. \tag{31}$$

4.34 Normal reaction between window and reinforcing member

The normal load per unit length between the window and the reinforcing member due to the load P is given by

$$N = \frac{P}{r_n}$$

$$= \frac{\sqrt{2}\;pa\left(\frac{x}{a}\right)^2}{\left\{1 + \left(\frac{x}{a}\right)^2\right\}^{\frac{1}{2}}} \tag{32}$$

and the window must therefore be designed to withstand these edge reactions and the normal pressure p. For complete neutrality the edge of the window should, under this sytem of loading, deform in the same manner as the reinforcing member.

4.35 Effect of superimposed shear stress

If there are shear stresses in the walls of the cylinder in the region of a proposed opening it is still theoretically possible to design a neutral hole provided the shear stress is always a constant proportion (say, K) of the hoop stress, a condition which does not occur in aircraft fuselages. Even so such a hole will seldom be feasible, for it is in the form of an ellipse with axes in the ratio

$$\frac{3 + \sqrt{(1 + 16K^2)}}{\sqrt{\{8\,(1 - 2K^2)\}}} \tag{33}$$

and inclined to the longitudinal axis at an angle

$$\tfrac{1}{2}\tan^{-1}(4K) \;. \tag{34}$$

For example, if K is $\frac{1}{4}$ the ratio of the axes is 1.67 and the angle of inclination is $22\frac{1}{2}^{\circ}$, but if K is $\frac{1}{2}$ the ratio of the axes goes up to 2.62.

If the shear stress is not a constant proportion of the hoop stress (or, indeed, for many cases where it is) it will be preferable to design the hole to be neutral under internal pressure and to reduce the inevitable stress concentrations due to shear by suitably increasing the wall thickness in the region of the hole. Note that such a scheme lends itself particularly to the cylinder with a row of openings in it, for a longitudinal strip of increased thickness in a cylinder does not cause any stress concentrations at the junction with the thinner wall of the rest of the cylinder (see para. 3.2).

5 Conical Shells

The membrane stresses are given by

$$\left.\begin{aligned} \sigma_\theta &= 2\sigma_z \\ &= \frac{p\ z \tan\beta}{t} \end{aligned}\right\} \qquad (35)$$

and if we design for a uniform membrane stress, we should have to take

$$t \propto z\ .$$

5.1 The shape of the neutral hole

The shape of the neutral hole in the developed surface of the conical shell, is no longer an exact ellipse, but is determined by the equation

$$z^3 - (3+\lambda^2)\ z\ Z^2 \cos\theta + 2Z^3(1-\lambda^2) = 0\ . \qquad (36)$$

As λ tends to zero, it can be shown that equation (36) represents an ellipse with axes in the ratio $\sqrt{2} : 1$.

5.2 The load in the reinforcing member

The load in the reinforcing member round the neutral hole is given by

$$P = \frac{p \tan\beta}{6}\left[3z^4 - 12z\,Z^3(1-\lambda^2) + Z^4(3+\lambda^2)^2\right]^{\frac{1}{2}}\ . \qquad (37)$$

6 Reinforcement on one side only of the shell wall

The analysis in paragraphs 3.1 and 4.32 referred to an idealised line reinforcement, but it is approximately valid if the c.g. of the reinforcement cross-section lies in the plane of the shell wall. If the reinforcement is eccentrically placed with respect to the shell wall some twisting of the reinforcement will take place under load and the reinforcement will not be so efficient. It is shown in Appendix VI that an eccentrically placed reinforcement of cross-sectional area $\bar{A}$ is approximately equivalent to centrally placed reinforcement of cross-sectional area A, where

$$A = \varepsilon\bar{A} \qquad (38)$$

and

$$\varepsilon = 1 - \frac{I_1^2}{I_o I_2} \qquad (39)$$

where

$$I_n = \iint y^n \, dS .$$

Some values of ε, which may be regarded as the efficiency of an eccentrically placed reinforcing member, are shown in Table I below when the reinforcement is on one side only of the shell wall.

Table I

Dependence of ε on the form of the reinforcement cross-section

Cross-sectional form of reinforcement	ε
rectangle	$\frac{1}{4}$
triangle	$1/3$
equal sided channel (U)	$\frac{1}{2}$
symmetrical angle (L)	$5/8$

The derivation of equation (39) is based on the assumption that the reinforcing member does not distort in the plane of its cross-section. With this assumption it is theoretically possible to obtain considerably higher values for ε. The problem is considered in further detail in Appendix VI.

For the problem considered in para. 3.1 the rotation of the cross-section of the reinforcing member is related to the radial displacement by the equation

$$\omega = \Delta \left(\frac{I_1}{I_2}\right) \qquad (40)$$

and there will be a localised bending of the wall of the shell to accommodate this rotation. The maximum stress in the wall due to this bending is given by

$$\sigma = 0.7 \, E \, \omega \left(\frac{t}{r}\right)^{\frac{1}{2}} \qquad (41)$$

7 Conclusions

This report considers the stresses developed under internal pressure in thin-walled shells of spherical, cylindrical or conical form. Formulae have been presented for predicting the maximum stresses in

(i) a spherical shell with

(a) a reinforced circular opening

(b) a cap of greater thickness;

and (ii) a cylindrical shell with

(c) Stringers and closely spaced frames

(d) Stringers and widely spaced frames

(e) End caps of various forms.

The feasibility of neutral holes in shells which have a developable surface has been ascertained, and investigated in detail for the cylindrical and conical shells.

A simple formula is given to allow for the reduced efficiency of reinforcement on one side only of the shell wall.

REFERENCES

No.	Author	Title, etc.
1	S. Timoshenko	Theory of Plates and Shells. McGraw Hill, 1940.
2	E.H. Mansfield	Neutral holes in plane sheet:- reinforced holes which are elastically equivalent to the uncut sheet. R & M 2815. September, 1950.
3	E.H. Mansfield	Elasticity of a sheet reinforced by stringers and skew ribs, with applications to swept wings. R & M 2758. November, 1949.

ADDITIONAL SYMBOLS USED ONLY IN THE APPENDICES

δ	=	radial movement in plane of ring
M_ϕ	=	bending moment per unit length in shell
λ	=	$\{3\,(1-\nu^2)(r/t)^2\}^{\frac{1}{4}}$
ψ	=	angle as defined in Chapter XII of Ref.1
M	=	bending moment per unit length between cap and sphere
V	=	rotation of cap or sphere along common boundary
χ_1, χ_2, χ_3		are defined in equation (53)
L	=	direct load in a frame
w	=	radial displacement (Appendix IV)
ϕ	=	force function defined in equation (58)
N_x, N_y, N_{xy}	=	direct and shear membrane forces per unit length
N_g, r_g		are defined after equation (57)
N_s, N_n, N_{sn}	=	direct and shear membrane forces per unit length
Os, On	=	axes tangential and normal to boundary
Σ	=	co-ordinate measured along boundary
Ω	=	angle between tangent to curved boundary and Os
ω	=	rotation of reinforcement cross-section
U	=	strain energy stored in complete ring
c, d	=	sides of rectangle
γ, ζ		are defined after equation (75)

APPENDIX I

Spherical shell with reinforced circular opening

The analysis given here is based on the simplified treatment given in Chapter XII of Ref.1.

Under purely membrane stresses the radial discrepancy, in the plane of the ring, between the ring and the shell is

$$\Delta_{ring} - \Delta_{shell}$$

and this must be eliminated by a radial force H per unit length. The radial movement of the ring and the shell due to H is

$$\left.\begin{aligned} \delta_{ring} &= \frac{Hr^2 \sin^2 \alpha}{EA} \\ \delta_{shell} &= \frac{2H \sin^2 \alpha}{E} \left(\frac{r}{t}\right)^{3/2} \{3\ (1-\nu^2)\}^{\frac{1}{4}} \end{aligned}\right\} \tag{42}$$

from equation (274) of Ref.1.

Equation (7) follows by equating

$$\Delta_{ring} - \Delta_{shell} = \delta_{ring} + \delta_{shell}. \tag{43}$$

In determining the effect of the H forces on the stresses in the shell wall it will be noticed from equation (10) that

$$\frac{H}{t} \text{ is of order } p\left(\frac{r}{t}\right)^{\frac{1}{2}}$$

which may be neglected in comparison with the membrane stresses in the shell wall (see equation (1)). The bending stresses may not be neglected, for we have from Ref.1:

$$M_\phi = \left(\frac{H \sin \alpha \sqrt{(rt)}}{\{3\ (1-\nu^2)\}^{\frac{1}{4}}}\right) e^{-\lambda\psi} \sin \lambda\psi \tag{44}$$

where

$$\lambda^4 = 3\ (1-\nu^2) \left(\frac{r}{t}\right)^2$$

Substituting the value of H from equation (10) in equation (44) and putting ν equal to 0.3, gives

$$M_\phi = 0.11\ prt \left(\frac{A^*}{A} - 1\right) e^{-\lambda\psi} \sin \lambda\psi$$

and the maximum value of $e^{-\lambda\psi} \sin \lambda\psi$ is

$$\frac{e^{-\pi/4}}{\sqrt{2}} = 0.32$$

which occurs when

$$\lambda\psi = \frac{\pi}{4} .$$

Now the maximum tensile stress in the shell wall is related to the bending moment by the equation

$$\sigma_{bending} = \frac{6}{t^2} \left| M_\phi \right| \tag{45}$$

and the corresponding stress concentration factor is therefore

$$1 + \frac{\sigma_{bending}}{\sigma_\phi} = 1 + 2 \times 6 \times 0.11 \times 0.32 \left| \frac{A^*}{A} - 1 \right|$$

c.f. equation (12).

APPENDIX II

Spherical shell with cap of greater thickness

If the parts are considered first to be acting as membranes and suffices 1 and 2 refer to the shell and cap respectively, we have

$$\left.\begin{aligned} \Delta_1 &= \frac{pr^2\,(1-\nu)\,\sin\alpha}{2Et} \\ &= n\,\Delta_2 \end{aligned}\right\} \qquad (46)$$

Furthermore, it will be noted that under the membrane forces there will be no change in the slopes of the shell wall and the cap. Because $\Delta_1 \neq \Delta_2$ there will be additional interactions (H and M) along the common boundary between cap and sphere.

The relations between the rotation V and the displacement δ and H and M are given approximately by equations (273) and (274) of Ref.1:

$$\left.\begin{aligned} V_1 &= \frac{4\lambda_1^3 M_1}{Ert} + \frac{2\lambda_1^2 \sin\alpha\, H_1}{Et} \\ \delta_1 &= \frac{2\lambda_1^2 \sin\alpha\, M_1}{Et} + \frac{2\lambda_1 r \sin^2\alpha\, H_1}{Et} \end{aligned}\right\} \qquad (47)$$

and if we express λ_2 in terms of λ_1:

$$\left.\begin{aligned} V_2 &= \frac{4\lambda_1^3 M_2}{Ert\,n^{2.5}} + \frac{2\lambda_1^2 \sin\alpha\, H_2}{Etn^2} \\ \delta_2 &= \frac{2\lambda_1^2 \sin\alpha\, M_2}{Etn^2} + \frac{2\lambda_1 r \sin^2\alpha\, H_2}{Etn^{1.5}} \end{aligned}\right\} \qquad (48)$$

The conditions of equilibrium and compatibility are

$$\left.\begin{aligned} H_1 &= -H_2 \\ M_1 &= M_2 = M, \text{ say} \\ V_1 &= -V_2 \\ \Delta_1 - \Delta_2 &= \delta_2 - \delta_1 \end{aligned}\right\} \qquad (49)$$

Equations (47), (48) and (49) may be solved to give

$$M = \frac{prt}{4}\left(\frac{1-\nu}{3\ (1+\nu)}\right)^{\frac{1}{2}}\left[\frac{n\ (n-1)(n^2-1)}{1 + 2\ (n^{1.5} + n^2 + n^{2.5}) + n^4}\right] \qquad (50)$$

and equation (13) is derived from equations (50) and (45) with ν equal to 0.3.

APPENDIX III

Shell reinforced by widely spaced frames (no stringers)

The problem is virtually solved in Chapter XI of Ref.1 where it is shown that the equations relating the load L in a frame and the longitudinal bending moment in the shell adjacent to the frame are

$$\frac{L\mu}{w}\left[\chi_1(\mu) - \frac{1}{2}\frac{\chi_2(\mu)}{\chi_3(\mu)}\right] = p\,(1-\tfrac{1}{2}\nu) - \frac{L}{Fw} \tag{51}$$

and

$$M = \frac{w^2\,\chi_2(\mu)}{2\,\mu^2}\left[p\,(1-\tfrac{1}{2}\nu) - \frac{L}{Fw}\right]$$

where

$$\left.\begin{aligned} \chi_1(\mu) &= \frac{\cosh\mu + \cos\mu}{\sinh\mu + \sin\mu} \\ \chi_2(\mu) &= \frac{\sinh\mu - \sin\mu}{\sinh\mu + \sin\mu} \\ \chi_3(\mu) &= \frac{\cosh\mu - \cos\mu}{\sinh\mu + \sin\mu} \end{aligned}\right\} \tag{53}$$

whence it will be seen that

$$\left[\chi_1(\mu) - \frac{1}{2}\frac{\chi_2(\mu)}{\chi_3(\mu)}\right] \text{ reduces to } \frac{\sinh\mu + \sin\mu}{2\,(\cosh\mu - \cos\mu)}$$

and the derivation of equation (19) is now straightforward.

APPENDIX IV

Shell reinforced by widely spaced frames (with stringers)

For the case without stringers considered in Appendix III the differential equation for the radial displacement w is given by equation (228) of Ref.1:

$$\left(\frac{Et^3}{12\ (1-\nu^2)}\right)\frac{d^4w}{dx^4}+\left(\frac{Et}{r^2}\right)w\ =\ 0\ . \qquad (54)$$

The equation is identical with that of a beam on an elastic foundation. When there are stringers present the equation takes the form

$$\left(\frac{EI_s}{b}\right)\ \frac{d^4w}{dx^4}+\left(\frac{Et}{r^2}\right)w\ =\ 0\ . \qquad (55)$$

The analysis for the two cases is then identical except for the substitution of η for μ .

APPENDIX V

Neutral holes in developable surfaces

Let us begin with a definition[2]: a neutral hole in a structure is a reinforced hole which is elastically equivalent to the uncut, or continuous, structure. The structure considered here is a shell whose walls are developable.

If the only body forces acting on the shell wall are pressure forces normal to the shell wall the equations of equilibrium of an element of the shell wall are

$$\left.\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} &= 0 \end{aligned}\right\} \qquad (56)$$

$$N_g = pr_g \qquad (57)$$

where Ox, Oy are Cartesian co-ordinates in the developed flat sheet and r_g is the local principal curvature of the surface (i.e. normal to the generator) and N_g is the load per unit length in the shell wall normal to the generator. (Compare, for example, with equation (224) of Ref.1 where a cylindrical surface is considered.)

Equation (56) is identical with the corresponding equations for a plane sheet and imply that there is a function ϕ from which the membrane forces may be determined as follows:

$$\left.\begin{aligned} N_x &= \frac{\partial^2\phi}{\partial y^2} \\ N_y &= \frac{\partial^2\phi}{\partial x^2} \\ N_{xy} &= -\frac{\partial^2\phi}{\partial x\partial y} \end{aligned}\right\} \qquad (58)$$

It will be noticed that the function ϕ is independent of the co-ordinate system: if fresh Cartesian axes Ox_1, Oy_1 are chosen we have

$$N_{x_1} = \frac{\partial^2\phi}{\partial y_1^2}\text{, etc.} \qquad (59)$$

Consider now the equilibrium conditions adjacent to a neutral reinforcement. Take Cartesian axes Os, On tangential and normal to the boundary, and let Σ be the distance along the curved boundary, and Ω be the angle between a tangent to the curved boundary and Os so that the curvature of the boundary in the developed flat sheet is $\frac{\partial\Omega}{\partial\Sigma}$. Resolving in the s- and n-directions gives

$$\frac{\partial P}{\partial \Sigma} + N_{sn} = 0 \tag{60}$$

and

$$P\frac{\partial\Omega}{\partial\Sigma} - N_n = 0 . \tag{61}$$

These equations may be put in terms of ϕ by virtue of equations (58) and (59):

$$\frac{\partial P}{\partial \Sigma} - \frac{\partial^2 \phi}{\partial s \partial n} = 0 \tag{62}$$

and

$$P\frac{\partial\Omega}{\partial\Sigma} - \frac{\partial^2 \phi}{\partial s^2} = 0 . \tag{63}$$

Now from geometrical considerations

$$\left.\begin{aligned} \frac{\partial\phi}{\partial s} &\equiv \frac{\partial\phi}{\partial\Sigma} \\ \frac{\partial^2\phi}{\partial s^2} &\equiv \frac{\partial^2\phi}{\partial\Sigma^2} + \frac{\partial\phi}{\partial n}\frac{\partial\Omega}{\partial\Sigma} \end{aligned}\right\} \tag{64}$$

whence equation (62) may be integrated to give

$$P = \frac{\partial\phi}{\partial n} \tag{65}$$

and equations (63), (64) and (65) reduce to

$$\frac{\partial^2\phi}{\partial\Sigma^2} = 0 \tag{66}$$

so that for a closed boundary we have

$$[\phi]_\Sigma = \text{constant.} \tag{67}$$

Equation (67) determines the shape of the hole boundary, and equation (65) then determines the load in the reinforcing member. The direct stiffness of the reinforcing member is determined from equation (65) and the known strain in the adjacent shell wall.

But these conditions alone do not suffice to determine a neutral hole. This is because appreciable bending of the reinforcing member will occur unless the unbalanced component of the force normal to the surface, due to the fact that the reinforcing member is curved, can be balanced. The balancing load required is obtained by resolving normal to the surface, whence

$$N = \frac{P}{r_n} . \tag{68}$$

Fortunately in pressurised shells the presence of stiff window frames ensures that equation (68) will be satisfied. In general, however, if there is no such supporting structure or external agency to fulfil equation (68) a neutral hole will not be feasible.

Example (Pressurised conical shell)

With the notation used in Section 5 the stresses in polar (z,θ) co-ordinates in the developed flat sheet are such that

$$\left.\begin{aligned} t\sigma_\theta &= 2t\sigma_z \\ &= (p \tan \beta)\; z, \\ t\tau_{\theta z} &= 0 \end{aligned}\right\} \tag{69}$$

whence

$$\phi \propto (z^3 + ax + by + c)$$

and equations (36) and (37) follow immediately.

APPENDIX VI

Efficiency of eccentrically placed reinforcement

In what follows the assumption will be made that the reinforcing member does not distort in the plane of its cross-section. Such an assumption will be justifiable for most practical reinforcements whose cross-section is sufficiently compact. The assumption will, of course, tend to overestimate the actual stiffness and efficiency of the reinforcement. We are only concerned with reinforcements which are neutral, or nearly so, and this sets a limit on the flexibility of the reinforcement we need consider; for such reinforcements the action of the sheet itself in resisting the rotation of the reinforcement cross-section is very small and will be ignored.

A further simplifying assumption will be made later that the width of the reinforcement is small in comparison with the radius of the hole, but this is not essential to the analysis.

Consider now an arbitrary cross-section of a circular ring attached to a plane sheet under uniform "hydrostatic" tension. We shall let the line of attachment between ring and sheet expand radially an amount Δ and determine the resultant rotation ω and hence the stiffness of the ring.

A typical point at (r, y) moves radially an amount $(\Delta - \omega y)$ and the circumferential strain is therefore

$$\left(\frac{\Delta - \omega y}{r}\right)$$

and the energy stored in the complete ring is therefore given by

$$U = \pi E \iint \left(\frac{\Delta - \omega y}{r}\right)^2 r \, dr \, dy$$

$$= \pi E \left[\Delta^2 \iint \frac{1}{r} \, drdy - 2\Delta\omega \iint \frac{y}{r} \, drdy + \omega^2 \iint \frac{y^2}{r} \, dr \, dy\right] \qquad \ldots\ldots(70)$$

Expression (70) must be minimised with respect to ω, whence

$$\frac{\omega}{\Delta} = \frac{\iint \frac{y}{r} \, drdy}{\iint \frac{y^2}{r} \, drdy} . \qquad (71)$$

If the width of the ring is small compared with the radius of the hole the factor $(1/r)$ in the various integrals will remain sensibly constant and we may write

$$\frac{\omega}{\Delta} = \frac{I_1}{I_2} \,. \qquad (40 \text{ bis})$$

With this simplification equation (70) becomes

$$U = \pi E \Delta^2 \left(I_o - \frac{I_1^2}{I_2} \right) \qquad (72)$$

c.f. $$\pi E \Delta^2 I_o$$

if there were no rotation of the cross-section.

The expression for the efficiency factor ε given in equation (39) follows from equation (72). If for example, the reinforcing ring is on one side only of the sheet and of rectangular cross-section ($c \times d$) it will be found that

$$\left.\begin{aligned} I_o &= cd \\ I_1 &= \tfrac{1}{2}\, cd^2 \\ I_2 &= (1/3)\, cd^3 \end{aligned}\right\} \qquad (73)$$

so that

$$\varepsilon = \tfrac{1}{4}.$$

But if the cross-section consists of two different, adjacent rectangles

$$\left.\begin{aligned} I_o &= c_1 d_1 + c_2 d_2 \\ I_1 &= \tfrac{1}{2}\,(c_1 d_1^2 + c_2 d_2^2) \\ I_2 &= (1/3)(c_1 d_1^3 + c_2 d_2^3) \end{aligned}\right\} \qquad (74)$$

and

$$\varepsilon = 1 - \frac{3\,(1+\gamma)^2}{4\,(1+\gamma\zeta)(1+\gamma/\zeta)} \qquad (75)$$

where

$$\gamma = c_2 d_2^2/(c_1 d_1^2)$$

$$\zeta = d_2/d_1$$

and it is clear that ε can be made to approach unity by letting γ remain finite while ζ becomes very large. Unfortunately these requirements are not compatible with the initial assumption that the reinforcement will not distort in the plane of its cross-section.

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