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The Technique of Flutter Calculations

By

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The Technique of Flutter Calculations

by

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SUMMARY

This Report describes the basic principles on which theoretical flutter analyses are made, and illustrates them by some simple applications. The techniques employed are typical of those in current use in this Country. Three Appendices give the two-dimensional aerodynamic derivatives for a wing-aileron-tab system, computational details of typical forms of solution, and an illustration of the use of resonance test modes in flutter calculations.

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1 Introduction

In 1948/49 a series of flutter courses was held at the R.A.E. for the purpose of introducing to technicians in the Aircraft Industry the methods used in making flutter calculations. A memorandum* was written at the time describing these methods. Although the memorandum was necessarily limited in scope, it has proved to be of considerable value as an introduction for those new to the subject and as a reference to the methods typically employed in flutter investigations. It has therefore been decided to re-issue it as a formal Report for general use. A few detailed improvements have been made, and parts of the original that related directly to the broader purpose of the flutter courses have been omitted.

This Report, as it stands, is wholly concerned with the techniques used in calculations for predicting theoretically the flutter characteristics of an aircraft. The techniques described are typical of those in current use in this Country. The Report, however, does no more than illustrate the basic principles involved. It does not give a realistic picture of the comprehensive nature of the flutter calculations normally required, nor does it describe all the detailed computational methods that may be employed.

The general basis of flutter calculations is first described. This is followed by two typical analyses illustrating the application of the basic analytical approach to the prediction of wing-aileron flutter and wing flexure-torsion flutter. In Appendix I expressions are given for the aerodynamic derivatives of a wing-aileron-tab system for two-dimensional incompressible flow. Appendix II amplifies the description given of the forms of solution in the typical analyses. Appendix III describes the interpretation and use of ground resonance test results, and illustrates the use of resonance test modes in flutter calculations by a typical calculation for fuselage-elevator flutter.

No attempt is made to provide a bibliography of flutter literature, which is not required for the restricted purpose of this note. The few references quoted are given as footnotes to the text.

Acknowledgements are made to Mr. E. G. Broadbent and Mr. W. G. Molyneux for the calculations of Appendix III and for their assistance in writing this Appendix.

2 The Basis of Flutter Calculations

The physical and mechanical aspects of flutter have been well described by earlier writers on the subject**, and it is not proposed to deal any further with these aspects here. Suffice it to say that the vibrating aeroplane is simply an elastic structure supporting certain masses (that is, having certain inertia properties) and subjected to aerodynamic forces of an oscillatory nature. There are therefore two main aspects to be considered: the elastic-inertia characteristics of the structure and the nature of the aerodynamic forces.

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* R.A.E. Technical Memo. Structures 8. "The Technique of Flutter Investigations."

** W.J. Duncan. "The Fundamentals of Flutter". R.A.E. Report No. Aero. 1920.

P.B. Walker. "The Mechanical Aspect of Flutter". Aircraft Engineering, February, 1938.

On the elastic-inertia side the problem may be considered in relation to that of the natural oscillations of the structure in vacuo, where each principal oscillation has a frequency and mode associated with it, the mode being the shape of the deformed state of the structure relative to its equilibrium position. For all but the simplest structures it is generally impossible to obtain an exact solution of the mode, except by iteration, and it is common practice to prescribe an arbitrary mode on the basis that if it is a reasonable approximation to the true mode then the frequency will not be unduly affected. Better still, a combination of several arbitrary modes may be prescribed, with amplitude ratios to be determined along with the frequency. The structure is then termed semi-rigid in the sense that it is allowed to deform only in a limited number of defined ways; or, in other words, it has a limited number of degrees of freedom. The term "degree of freedom" is fairly self-explanatory but for the sake of clarity may be defined as a prescribed deformation or movement of the structure whose amplitude in relation to that of any other degree of freedom is not assumed but remains to be determined. The mode associated with a degree of freedom may conveniently be termed the freedom mode.

In flutter the structure is likewise treated as semi-rigid and the major problem on the elastic-inertia side is to know how many and what sort of degrees of freedom to consider in order to provide a satisfactory representation of the true mode in the critical flutter condition. For a complete flutter investigation on any particular aeroplane the number of degrees of freedom considered should be large enough to cover all possible deformations of the various components as well as control surface movements and bodily movements of the aeroplane as a whole. Assuming that such a process were practicable, it would even so be found generally that in any critical condition a certain few of the degrees of freedom predominated, their amplitudes being much greater than those of the remainder: the resulting flutter would then be designated as being of a particular "type", involving those components associated with the major degrees of freedom. Wing flexure - aileron flutter for instance is the flutter which arises when the wing flexural mode and aileron rotation predominate: fuselage bending and elevator rotation would similarly result in fuselage - elevator flutter. It is therefore generally possible to investigate any particular type of flutter with a relatively small number of degrees of freedom. For so-called "classical" flutter, with which this Report is concerned, at least two degrees of freedom must be present: although each degree of freedom would separately give a damped oscillation the various couplings that exist between the two can result in an unstable oscillation when combined together under certain conditions. The labour involved in a flutter calculation increases greatly with the number of degrees of freedom, and for the average routine investigation the practical limit is set at four. For most routine work, however, two to four degrees of freedom are generally adequate.

On the aerodynamic side the forces are expressed in the form of derivatives, which define the amount of the particular force concerned per unit displacement, velocity, or acceleration of the particular motion concerned, the motion being relative to the equilibrium position. The aerodynamic derivatives used in flutter prediction are mainly theoretical values based on the following assumptions:-

(a) thin aerofoil theory

(b) perfect fluid with two-dimensional irrotational flow

(c) simple harmonic motion of the surfaces.

A complete list of two-dimensional incompressible flow derivatives is given in Appendix I* for a wing-aileron-tab combination: it is equally applicable of course to a tailplane-elevator-tab system. Motion of the system is represented by the displacement of some reference point on the wing chord together with rotations of wing, aileron, and tab about the reference point and hinge positions respectively. In Appendix I the leading edge is used as reference point and the derivatives are termed "leading edge" derivatives. The form of the expressions for the aerodynamic forces is explained in Section 3. It is to be noted that the damping and stiffness derivatives (such as $\ell_{\dot{z}}$ and ℓ_z) which relate to velocity and displacement are functions of the frequency.

To use two-dimensional derivatives as they stand would be tantamount to assuming that the aerodynamic forces on any chordwise strip of the wing are the same as if the strip were part of a uniform wing of infinite span undergoing the same motion as the strip. For practical wings such an assumption is of course not justified, and it is usual to apply approximate correction factors to the two-dimensional derivatives, based on the known values of the static derivatives (a_1, a_2, b_1, b_2, etc.) for the complete three-dimensional wing. For wings of low aspect ratio more accurate values are required, and experimental and theoretical work is in hand to this end.

Elastic-inertia and aerodynamic effects are combined in a flutter calculation by straightforward application of the Lagrangian equations of motion for a non-conservative system to the critical flutter condition in which the motion is simple harmonic, representing transition from a decaying to a growing oscillation. Typical ternary and binary analyses involving three and two degrees of freedom are given in detail in Sections 3 and 4 respectively. Simple uncoupled freedom modes are used for the wing deformation in these analyses, one of pure flexure and one of pure torsion. Modes of this type are often termed "arbitrary" modes in contrast to the normal modes associated with the natural oscillations in vacuo or in still air, which as discussed later may also be used for the freedom modes: in actual fact of course any freedom modes used with semi-rigid structures are essentially arbitrary. The distinction has arisen because in many cases normal modes do provide a better approximation to the flutter mode than do the simple arbitrary modes, and also because they provide a stiffness representation that is more accurately related to the freedom mode.

The ternary analysis is given first, from which the binary analysis in Section 4 follows very simply by making the omissions appropriate to the deleted degree of freedom. This procedure is adopted purposely in preference to a detailed binary analysis followed by a rather complicated presentation of the effect of introducing a third degree of freedom.

An unswept wing is assumed in the analyses, in which it will be noted that the flexural axis is taken as the reference axis for the wing motion, involving a transformation of the leading-edge derivatives. If the analysis is applied to an ad hoc calculation, which is primarily the intention, then the unknowns are the frequency of the oscillation and the airspeed in the critical flutter condition. The solution of the equations of motion is complicated by the dependency of the damping and stiffness derivatives on frequency. General forms of solution described in Sections 3 and 4 are given in greater detail in Appendix II. It should, incidentally, be mentioned that the notation used for the typical analyses and throughout

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* Values for a wider range of tab chord ratios are given by Minhinnick in R.A.E. Report No. Structures 86. Theoretical values of two-dimensional subsonic compressible flow derivatives are given by Minhinnick in R.A.E. Report No. Structures 87.

the report generally is by no means universal: various systems of notation are used by different workers, so much so in fact that serious consideration is being given to the possible adoption of a universal system. At the moment, however, the notation used in this report will be found adequate for the immediate purpose.

Finally, in Appendix III, an outline is given of the usefulness of resonance tests in flutter investigations. The analysis of resonance tests is by no means a cut-and-dried science, being still at the stage where knowledge grows with experience. Resonance tests have, however, more than once indicated possibly dangerous modes conducive to flutter and have thereby enabled preventive measures to be taken in time. They are particularly useful, of course, in cases where no specific theoretical flutter investigations have been made in the design stage and reliance has been placed on the standard stiffness and inertia criteria, which do not pretend to cover all eventualities. Any flutter calculations made as a result of resonance tests will generally use the resonance modes, which will be the normal modes of vibration as distinct from "arbitrary" modes. This makes no difference to the form of the analysis: a binary calculation similar to that presented in Section 4 might for instance be done either as given there using two arbitrary modes, one of pure flexure and one of pure torsion; or it might be done using two normal modes, each of which would involve both flexure and torsion. There are certain advantages in using normal modes, which may, resonance tests apart, be sufficient in some cases to warrant a theoretical estimation of such modes for use in a flutter calculation. In view of the interest attached to normal mode calculations, a typical investigation (in this case of fuselage - elevator flutter) is given at the end of Appendix III.

3 Typical Ternary Analysis. Wing Flexure and Torsion with Free Aileron

The case envisaged is that of the wing oscillating in flexure and in torsion, together with accompanying oscillation of the unconstrained aileron. The wing motion, like that of the aileron, is antisymmetric. Fuselage immobility is assumed, or in other words there is no rolling motion of the aeroplane as a whole, so that the wing motion is due entirely to structural distortion. Fuselage mobility could be included as an extra degree of freedom, making the calculation a quaternary one.

The analysis is based on the application of the standard Lagrangian equations to the case of the wing and aileron in the critical flutter condition, oscillating with constant amplitude or simple harmonic motion.

The Lagrangian equations are a statement of the energy relationships of a dynamical system whose configuration in space is determined or can be expressed by a number of so-called "generalised" co-ordinates q_1, q_2, etc. In the simple case of a rigid body with a single translational degree of freedom the equations reduce to the well known Force = Mass × Acceleration.

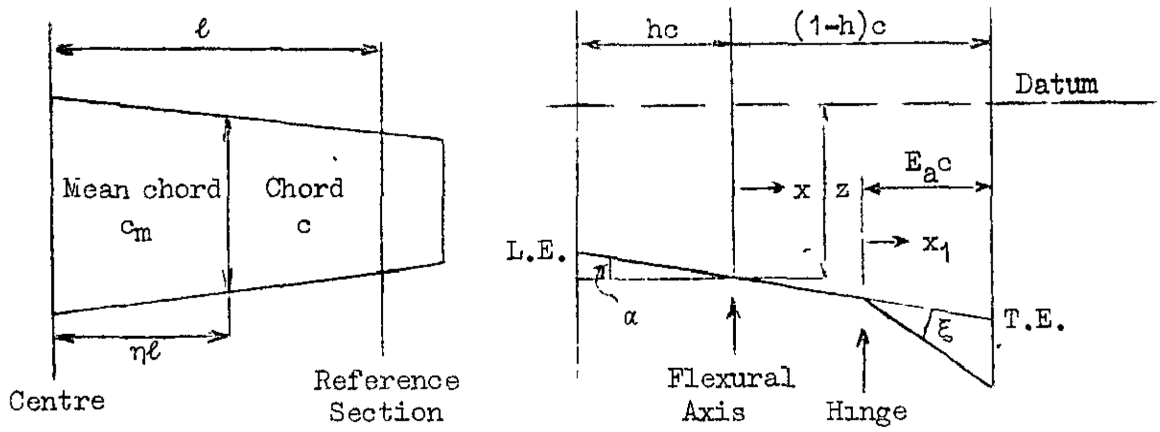
In general, the Lagrangian equation appropriate to the co-ordinate q_r is

$$\frac{d}{dt} \cdot \frac{\partial T}{\partial \dot{q}_r} + \frac{\partial V_e}{\partial q_r} = Q_r \qquad (1.1)$$

where T and V_e are the kinetic and potential energies of the system and Q_r is the "generalised" force appropriate to the co-ordinate q_r (see later). Strictly speaking a further term $-\frac{\partial T}{\partial q_r}$ should be included in equation (1.1) but as small displacements are assumed for which the kinetic energy is a function only of the velocities $\dot{q}_r$ and not of the displacements q_r the term is here omitted.

To apply the equations to the critical flutter condition the wing-aileron motion is represented by conveniently chosen co-ordinates and the various terms in the Lagrangian equations evaluated in order.

Wing-Aileron Motion (Arbitrary Modes)



The flexural axis is chosen as the axis of reference and the wing motion represented by a downward displacement z of this axis and a nose-up rotation α about the axis, both relative to the equilibrium position. Arbitrary modes

$$z = z_o \cdot f(\eta)$$

$$\alpha = \alpha_o \cdot F(\eta) \qquad (1.2)$$

are chosen, z_o and α_o being the values of z and α at the reference section, where η and the displacement functions f and F are all unity. Co-ordinates q_1 and q_2 are then chosen to represent the displacements z_o and α_o at the reference section as follows:

$$q_1 = \frac{z_o}{\ell}$$

$$q_2 = \frac{c_m}{\ell} \cdot \alpha_o \qquad (1.3)$$

Combining (1.2) and (1.3) gives the wing motion in terms of the generalised co-ordinates as

$$z = \ell f \cdot q_1$$

$$\alpha = \frac{\ell}{c_m} F \cdot q_2 \qquad (1.4)$$

For the aileron motion the angle relative to the wing is likewise specified by the angle at the reference section, ξ_0. If, as is quite common, the aileron is assumed rigid torsionally, then the local aileron angle is given by

$$\xi = \xi_o + \alpha_o - \alpha \tag{1.5}$$

If ξ_0 is represented directly by a third co-ordinate $q_3 = \frac{c_m}{\ell} \cdot \xi_o$, (1.5) becomes

$$\xi = \frac{\ell}{c_m} \cdot q_3 + \frac{\ell}{c_m}(1 - F) \cdot q_2 \tag{1.6}$$

The aileron mode is thus a function of two of the three co-ordinates.

In the critical flutter condition the displacements z_0, α_0, ξ_0, and therefore the corresponding co-ordinates q_1, q_2, q_3, vary sinusoidally with time. If $\frac{p}{2\pi}$ is the frequency of the oscillation in cycles per second, then

$$\frac{\ddot{q}_1}{q_1} = \frac{\ddot{q}_2}{q_2} = \frac{\ddot{q}_3}{q_3} = -p^2 = -\omega_m^2 \cdot \frac{V^2}{c_m^2} \tag{1.7}$$

ω is the local frequency parameter $\frac{pc}{V}$, and ω_m the mean frequency parameter corresponding to the mean chord c_m. V is the airspeed.

<u>Inertia Coefficients</u> $\left(\text{from term } \frac{d}{dt} \cdot \frac{\partial T}{\partial \dot{q}_r}\right)$

For an element of mass δm situated in the wing a distance x behind the reference axis, the downward velocity is $(\dot{z} + x\dot{\alpha})$. For a similar mass in the aileron a distance x_1 behind the hinge the velocity is $(\dot{z} + x\dot{\alpha} + x_1\dot{\xi})$.

The total kinetic energy for the half-wing is then

$$T = \sum_{\text{wing}} \tfrac{1}{2} (\dot{z} + x\dot{\alpha})^2 \, \delta m + \sum_{\text{aileron}} \tfrac{1}{2} (\dot{z} + x\dot{\alpha} + x_1\dot{\xi})^2 \, \delta m \tag{1.8}$$

Substituting for z, α, and ξ from (1.4) and (1.6) gives

$$T = \sum_{\text{wing}} \tfrac{1}{2} \left(\ell f \dot{q}_1 + x \frac{\ell}{c_m} F \dot{q}_2\right)^2 \delta m$$

$$+ \sum_{\text{aileron}} \tfrac{1}{2} \left(\ell f \dot{q}_1 + \frac{\ell}{c_m} \{xF + x_1 (1 - F)\} \dot{q}_2 + x_1 \frac{\ell}{c_m} \dot{q}_3\right)^2 \delta m \tag{1.9}$$

that is, a function of the three co-ordinate velocities $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$.

For the equation in q_1 the appropriate inertia term is

$$\frac{d}{dt} \cdot \frac{\partial T}{\partial \dot{q}_1} = \sum_{\text{wing}} \left(\ell f \ddot{q}_1 + x \frac{\ell}{c_m} F \ddot{q}_2\right) \ell f \,.\, \delta m$$

$$+ \sum_{\text{aileron}} \left(\ell f \ddot{q}_1 + \frac{\ell}{c_m} \{xF + x_1 (1 - F)\} \ddot{q}_2 + x_1 \frac{\ell}{c_m} \ddot{q}_3\right) \ell f \,.\, \delta m$$

$$= - \rho \ell^3 V^2 (a_{11} q_1 + a_{12} q_2 + a_{13} q_3) \omega_m^2 \qquad (1.10)$$

where, by using equation (1.7), the non-dimensional inertia coefficients are obtained as

$$a_{11} = \frac{1}{\rho c_m^2} \int f^2 m \,.\, d\eta$$

$$a_{12} = \frac{1}{\rho c_m^3} \left[\int fF m \bar{x} \,.\, d\eta + \int f(1 - F) m \bar{x}_1 \,.\, d\eta\right]$$

$$a_{13} = \frac{1}{\rho c_m^3} \int f m \bar{x}_1 \,.\, d\eta$$

m is the mass per unit span (including the aileron), $m \bar{x}$ the mass moment about the reference axis per unit span (including the aileron), and $m \bar{x}_1$ the mass moment about the hinge per unit span (aileron only).

Similarly, for the equations in q_2 and q_3 the appropriate inertia terms are

$$\frac{d}{dt} \cdot \frac{\partial T}{\partial \dot{q}_2} = - \rho \ell^3 V^2 (a_{21} q_1 + a_{22} q_2 + a_{23} q_3) \omega_m^2 \qquad (1.11)$$

$$\frac{d}{dt} \cdot \frac{\partial T}{\partial \dot{q}_3} = - \rho \ell^3 V^2 (a_{31} q_1 + a_{32} q_2 + a_{33} q_3) \omega_m^2 \qquad (1.12)$$

with inertia coefficients

$$a_{21} = a_{12}$$

$$a_{22} = \frac{1}{\rho c_m^4} \left[\int F^2 mK^2 \,.\, d\eta + 2 \int F (1 - F) mK_2^2 \,.\, d\eta + \int (1 - F)^2 mK_1^2 \,.\, d\eta\right]$$

$$a_{23} = \frac{1}{\rho c_m^4} \left[\int F mK_2^2 \,.\, d\eta + \int (1 - F) mK_1^2 \,.\, d\eta\right]$$

$$a_{31} = a_{13}$$

$$a_{32} = a_{23}$$

$$a_{33} = \frac{1}{\rho c_m^4} \int mK_1^2 \, . \, d\eta$$

mK^2 is the mass moment of inertia about the reference axis per unit span (including the aileron), mK_1^2 the mass moment of inertia about the hinge per unit span (aileron only), and mK_2^2 the mass product of inertia $\Sigma\, x\, x_1\, \delta m$ about the reference axis and hinge per unit span (aileron only).

<u>Stiffness Coefficients</u> $\left(\text{from term } \frac{\partial V_e}{\partial q_r}\right)$

The potential energy stored in the wing during displacements z_o and α_o is equal to the work that would be done by any system of statically applied loads which produced the same wing deformation. By the semi-rigid principle this is equated to the work done by concentrated loads which, applied at the reference section, produce the same displacements z_o and α_o at the reference section.

In terms of the standard flexural and torsional stiffnesses ℓ_ϕ and m_θ appropriate to the reference section, the potential energy stored in the half-wing can then be expressed as

$$V_e = \tfrac{1}{2}\, \ell_\phi \left(\frac{z_o}{\ell}\right)^2 + \tfrac{1}{2}\, m_\theta\, \alpha_o^2$$

or, substituting for z_o and α_o from (1.3)

$$V_e = \tfrac{1}{2}\, \ell_\phi\, q_1^2 + \tfrac{1}{2}\, m_\theta \frac{\ell^2}{c_m^2} q_2^2 \qquad (1.13)$$

Since the aileron is unconstrained there is no elastic stiffness associated with it and consequently no additional energy stored in respect of the aileron motion. Equation (1.13) therefore gives the whole of the potential energy stored in the wing-aileron system.

For the equations in q_1 and q_2 the appropriate stiffness terms are

$$\frac{\partial V_e}{\partial q_1} = \ell_\phi\, q_1 = \rho\, \ell^3\, V^2 \,.\, e_{11} q_1 \qquad (1.14)$$

$$\frac{\partial V_e}{\partial q_2} = m_\theta \frac{\ell^2}{c_m^2} \cdot q_2 = \rho\, \ell^3\, V^2 \,.\, e_{22} q_2 \qquad (1.15)$$

from which the non-dimensional stiffness coefficients are obtained as

$$e_{11} = \frac{\ell_\phi}{\rho\ \ell^3 V^2} \qquad (1.16)$$

$$e_{22} = \frac{m_\theta}{\rho\ \ell^3 V^2} \cdot \frac{\ell^2}{c_m^2} = \frac{e_{11}}{R} \qquad (1.17)$$

where R, the stiffness ratio $= \frac{\ell_\phi}{m_\theta} \cdot \frac{c_m^2}{\ell^2}$.

For the equation in q_3 the stiffness term $\frac{\partial V_e}{\partial q_3}$ is zero. It should be noted that it is possible, as with the inertia coefficients, to have a total of nine stiffness coefficients. Cross stiffness coefficients e_{rs} $(r \neq s)$ are however eliminated by the choice of flexural axis as the wing reference axis and by the absence of interaction between wing and aileron motions. The direct stiffness coefficient e_{33} associated with the aileron motion is zero in this case, but would not be so, of course, if the aileron were constrained by holding the stick.

Aerodynamic Force Coefficients (from term Q_r)

The "generalised" force Q_r represents the externally applied loads appropriate to the co-ordinate q_r and is defined as follows. If due to a small displacement δq_r the work done by the applied loads is δW_r, then the generalised force is $Q_r = \frac{\delta W_r}{\delta q_r}$.

The applied loads in this case are the aerodynamic loads which on an oscillating aerofoil consist of contributions due to inertia, damping, and stiffness. Using derivatives appropriate to the reference axis, the lift force L, for instance, is per unit span,

$$\frac{L}{\rho c V^2} = (-\omega^2 \ell_{\ddot z} + i\,\omega \ell_{\dot z} + \ell_z)\frac{z}{c} + (-\omega^2\ \ell_{\ddot\alpha} + i\,\omega \ell_{\dot\alpha} + \ell_\alpha)\ \alpha$$
$$+ (-\omega^2\ \ell_{\ddot\xi} + i\,\omega \ell_{\dot\xi} + \ell_\xi)\ \xi$$

The three major terms in z, α, and ξ represent the contributions due to these three constituent motions and the interpretation of the form of these terms can be illustrated by the first, the term in z. Since z is varying sinusoidally with time at a frequency of $\frac{p}{2\pi}$ cycles per second, or in exponential form is proportional to e^{ipt}, then velocity and acceleration are respectively equal to ipz and $-p^2 z$, and therefore proportional to $i\omega z$ and $-\omega^2 z$, ω being the frequency parameter $\frac{pc}{V}$. The three terms in the bracket therefore represent in order the lift due to translational acceleration (inertia), velocity (damping), and displacement (stiffness), with the appropriate derivatives $\ell_{\ddot z}$ $\ell_{\dot z}$ and ℓ_z. There is in the present case a total of 27 derivatives, comprising inertia, damping, and stiffness derivatives for each of the three relevant forces in respect of each of the three displacements.

The three forces concerned, that is the lift L, moment about the reference axis M, and hinge moment H, may conveniently be written in the following shortened notation as

$$\left.\begin{aligned}
\frac{L}{\rho\, c V^2} &= L_z \cdot \frac{z}{c} + L_\alpha \cdot \alpha + L_\xi \cdot \xi \\
\frac{M}{\rho\, c^2 V^2} &= M_z \cdot \frac{z}{c} + M_\alpha \cdot \alpha + M_\xi \cdot \xi \\
\frac{H}{\rho\, c^2 V^2} &= H_z \cdot \frac{z}{c} + H_\alpha \cdot \alpha + H_\xi \cdot \xi
\end{aligned}\right\} \qquad (1.18)$$

where the complex derivative $L_z = -\omega^2\, \ell_{\ddot{z}} + i\,\omega\, \ell_{\dot{z}} + \ell_z$, and similarly for the other complex derivatives.

The work done by the aerodynamic forces on a unit span strip during displacements δz, $\delta\alpha$, and $\delta\xi$ is

$$\delta W = -L \cdot \delta z + M \cdot \delta\alpha + H \cdot \delta\xi$$

and by substituting (1.4), (1.6) this is obtained in terms of the co-ordinate increments as

$$\delta W = -\ell f\, L \cdot \delta q_1 + \frac{\ell}{c_m} F\, M' \cdot \delta q_2 + \frac{\ell}{c_m} H \cdot \delta q_3 \qquad (1.19)$$

where $M' = M + \frac{1 - F}{F} \cdot H$. (1.20)

Integrating spanwise over the wing gives the total work done during increments in q_1, q_2, and q_3. The "generalised" forces appropriate to the three co-ordinates then follow automatically by definition as

$$\left.\begin{aligned}
Q_1 &= \frac{\delta W_1}{\delta q_1} = -\int \ell^2 f\, L \cdot d\eta \\
Q_2 &= \frac{\delta W_2}{\delta q_2} = \int \frac{\ell^2}{c_m} F\, M' \cdot d\eta \\
Q_3 &= \frac{\delta W_3}{\delta q_3} = \int \frac{\ell^2}{c_m} H \cdot d\eta
\end{aligned}\right\} \qquad (1.21)$$

It remains only to substitute for L, M', and H and by expansion to obtain expressions for the three generalised forces in terms of the co-ordinates q_1, q_2, and q_3.

The first step is to obtain expressions for L, M', and H in terms of q_1, q_2, q_3.

Substituting for z, α, and ξ from (1.4), (1.6) in equation (1.18) for L gives finally

$$\frac{L}{\rho cV^2} = L_z \frac{\ell}{c} fq_1 + L_\alpha' \frac{\ell}{c_m} Fq_2 + L_\xi \cdot \frac{\ell}{c_m} q_3 \qquad (1.22)$$

where

$$L_\alpha' = L_\alpha + \frac{1-F}{F} L_\xi \qquad (1.23a)$$

and in the same way exactly similar expressions are obtained for $\frac{M}{\rho c^2V^2}$ and $\frac{H}{\rho c^2V^2}$ with compound-complex derivatives

$$M_\alpha' = M_\alpha + \frac{1-F}{F} M_\xi \qquad (1.23b)$$

and

$$H_\alpha' = H_\alpha + \frac{1-F}{F} H_\xi \qquad (1.23c)$$

Substituting these new forms for M and H in equation (1.20) then gives

$$\frac{M'}{\rho c^2V^2} = M_z' \cdot \frac{\ell}{c} fq_1 + M_\alpha'' \cdot \frac{\ell}{c_m} Fq_2 + M_\xi' \cdot \frac{\ell}{c_m} \cdot q_3 \qquad (1.24)$$

where

$$M_z' = M_z + \frac{1-F}{F} H_z \qquad (1.24a)$$

$$M_\alpha'' = M_\alpha' + \frac{1-F}{F} H_\alpha' \qquad (1.24b)$$

$$M_\xi' = M_\xi + \frac{1-F}{F} H_\xi \qquad (1.24c)$$

The second and final step is to substitute the new expressions for L, M', and H in equations (1.21) to obtain the generalised forces. Since L, M' and H have been reduced to linear functions of the co-ordinates q_1, q_2, and q_3 the generalised forces will also be obtained in this form. The coefficients of q_1, q_2, and q_3 in L, M' and H in every case include a complex derivative and the corresponding coefficients in the generalised forces will therefore consist of inertia, damping, and stiffness contributions. This can be illustrated by considering the coefficient of q_1 in the expression for the force Q_1. By substituting (1.22) in the first of equations (1.21) it is seen that the term in q_1 is

$$-\int \ell^2 f \cdot \rho cV^2 \cdot L_z \cdot \frac{\ell}{c} fq_1 \cdot d\eta = -\rho \ell^3 V^2 \int f^2(-\omega^2 \ell_{\ddot{z}} + i\omega \ell_{\dot{z}} + \ell_z)\, d\eta \cdot q_1$$

replacing the complex derivative L_z by its basic derivative form. The term in q_1 can then be written as

$$- \rho \ell^3 V^2 \left(-\gamma_{11} \omega_m{}^2 + b_{11}\, i\omega_m + c_{11}\right) . q_1$$

the mean frequency parameter ω_m replacing the local and variable parameter ω. The γ, b, and c coefficients represent respectively the inertia, damping, and stiffness contributions to the generalised force in respect of the particular co-ordinate, in this case q_1. The suffix notation employed for the coefficients is similar to that used for the structural inertia and elastic stiffness coefficients, the numbers 'rs' after a coefficient signifying the contribution to the force Q_r in respect of the co-ordinate q_s.

The three generalised forces can therefore be written generally as

$$\frac{Q_r}{\rho \ell^3 V^2} = - \sum_{s=1}^{3} \left(- \gamma_{rs}\ \omega_m{}^2 + b_{rs}\, i\omega_m + c_{rs}\right) q_s \qquad (1.25)$$

with r having values 1, 2, and 3. The total number of coefficients is thus 27 and their values are found by equating corresponding terms in equations (1.21) and (1.25).

A point to note here is that any compound complex derivative is conveniently expressible in terms of inertia, damping, and stiffness contributions involving appropriate derivatives. For instance, (1.23a) can be written

$$L_\alpha{}' = -\omega^2\, \ell_{\ddot{\alpha}}{}' + i\,\omega\, \ell_{\dot{\alpha}}{}' + \ell_\alpha{}'$$

where
$$\ell_\alpha{}' = \ell_\alpha + \frac{1 - F}{F}\, \ell_\xi$$

and similarly for the compound derivatives $\ell_{\dot{\alpha}}{}'$ and $\ell_{\ddot{\alpha}}{}'$. In other words, the compound inertia, damping, and stiffness derivatives are obtained from expressions exactly similar in form to those for the corresponding compound-complex derivatives. To illustrate the point further, the damping derivative $h_{\dot{\alpha}}{}'$ is, from (1.23c),

$$h_{\dot{\alpha}}{}' = h_{\dot{\alpha}} + \frac{1 - F}{F}\, h_{\dot{\xi}}$$

and the inertia derivative $m_{\ddot{\alpha}}{}''$ is, from (1.24b),

$$m_{\ddot{\alpha}}{}'' = m_{\ddot{\alpha}}{}' + \frac{1 - F}{F}\, h_{\ddot{\alpha}}{}'$$

It is not necessary to write down the complete list of 27 force coefficients since for any given order there is a simple relationship between the γ, b, and c coefficients of that order. The nine c coefficients are as follows:

$$c_{11} = \int f^2\, \ell_z \, . \, d\eta$$

$$c_{12} = \int \frac{c}{c_m} f F \ell_\alpha' \, . \, d\eta$$

$$c_{13} = \int \frac{c}{c_m} f \ell_\xi \, . \, d\eta$$

$$c_{21} = - \int \frac{c}{c_m} f F m_z' \, . \, d\eta$$

$$c_{22} = - \int \frac{c^2}{c_m^2} F^2 m_\alpha'' \, . \, d\eta$$

$$c_{23} = - \int \frac{c^2}{c_m^2} F m_\xi' \, . \, d\eta$$

$$c_{31} = - \int \frac{c}{c_m} f h_z \, . \, d\eta$$

$$c_{32} = - \int \frac{c^2}{c_m^2} F h_\alpha' \, . \, d\eta$$

$$c_{33} = - \int \frac{c^2}{c_m^2} h_\xi \, . \, d\eta$$

The b and γ coefficients for any given order are obtained from the c coefficient of the same order by including additional factors $\frac{c}{c_m}$ and $\frac{c^2}{c_m^2}$ within the integral and using the appropriate damping and inertia derivatives. For example, for the order '12' the coefficients are:

$$c_{12} = \int \frac{c}{c_m} f F \ell_\alpha' \, . \, d\eta$$

$$b_{12} = \int \frac{c^2}{c_m^2} f F \ell_{\dot\alpha}' \, . \, d\eta$$

$$\gamma_{12} = \int \frac{c^3}{c_m^3} f F \ell_{\ddot\alpha}' \, . \, d\eta$$

To evaluate the force coefficients the basic derivatives are first calculated. A complete list of these is given in Appendix I*, where it

*The basic derivatives given in Appendix I are two-dimensional derivatives appropriate to the leading edge as reference axis. Transformation formulae are also given from which corresponding derivatives may be obtained for other reference axes.

will be seen that the damping and stiffness derivatives are functions of the local frequency parameter ω and can therefore only be calculated for a given value of the mean frequency parameter ω_m, from which the local value is obtained as $\omega = \omega_m \cdot \frac{c}{c_m}$. From the basic derivatives follow the compound derivatives and finally, by spanwise integration over the half-wing, the coefficients themselves. Since the b and c coefficients are derived from damping and stiffness derivatives respectively it follows that they also must depend upon the value of ω_m.

Solution of the Equations of Motion

The Lagrangian equations can now be written down directly from the general equation (1.1) with r successively equal to 1, 2, and 3. The first equation with $r = 1$ is for instance obtained by substituting (1.10) for the inertia term, (1.14) for the stiffness term, and (1.25) with $r = 1$ for the force term, and the resulting equation is

$$\left\{- (a_{11} + \gamma_{11})\ \omega_m^{\ 2} + b_{11} i\omega_m + c_{11} + e_{11}\right\} q_1$$

$$+ \left\{- (a_{12} + \gamma_{12})\ \omega_m^{\ 2} + b_{12} i\omega_m + c_{12}\right\} q_2$$

$$+ \left\{- (a_{13} + \gamma_{13})\ \omega_m^{\ 2} + b_{13} i\omega_m + c_{13}\right\} q_3 = 0$$

By a similar process the two remaining equations are obtained. The complete set of equations may conveniently be written as

$$\left.\begin{aligned} (\delta_{11} + e_{11})\ q_1 + \delta_{12}\ q_2 + \delta_{13}\ q_3 &= 0 \\ \delta_{21}\ q_1 + (\delta_{22} + e_{22})\ q_2 + \delta_{23}\ q_3 &= 0 \\ \delta_{31}\ q_1 + \delta_{32}\ q_2 + \delta_{33}\ q_3 &= 0 \end{aligned}\right\} \qquad (1.26)$$

where in general

$$\delta_{rs} = - (a_{rs} + \gamma_{rs})\ \omega_m^{\ 2} + b_{rs}\ i\omega_m + c_{rs}$$

or

$$\delta_{rs} = (a_{rs} + \gamma_{rs})\ \lambda^2 + b_{rs}\ \lambda + c_{rs}\ ,$$

the symbol λ being used for the imaginary quantity $i\omega_m$.

Equations (1.26) are the equations of motion for the critical flutter condition, linear in q_1, q_2, and q_3, with complex coefficients. Eliminating q_1, q_2, and q_3 gives the determinantal equation

$$\begin{vmatrix} \delta_{11} + e_{11}, & \delta_{12}, & \delta_{13} \\ \delta_{21}, & \delta_{22} + e_{22}, & \delta_{23} \\ \delta_{31}, & \delta_{32}, & \delta_{33} \end{vmatrix} = 0 \qquad (1.27)$$

which is the relationship that must exist between the coefficients in the critical condition. Speed and frequency are the variables in (1.27), which when real and imaginary parts are equated to zero provides two equations for the solution of these quantities; but the solution is complicated by the form in which the variables occur. Excluding compressibility effects, speed occurs only in the stiffness coefficients e_{11} and e_{22}. The frequency parameter ω_m however occurs explicitly in the form of the δ coefficients and implicitly in the values of the force coefficients b and c. There are two possible methods of solution.

In the first, which might be termed the "direct iterative" method, (1.27) is expanded in the form of a polynomial in λ to the sixth power

$$p_o \lambda^6 + p_1 \lambda^5 + p_2 \lambda^4 + p_3 \lambda^3 + p_4 \lambda^2 + p_5 \lambda + p_6 = 0 \qquad (1.28)$$

where the coefficients p_o to p_6 are real functions of the original inertia, stiffness, and force coefficients. Equating to zero the real and imaginary parts of (1.28), involving even and odd powers of λ respectively, then gives the two real equations

$$\left.\begin{aligned} - p_o \omega_m^6 + p_2 \omega_m^4 - p_4 \omega_m^2 + p_6 &= 0 \\ p_1 \omega_m^4 - p_3 \omega_m^2 + p_5 &= 0 \end{aligned}\right\} \qquad (1.29)$$

from which ω_m can be eliminated to give an equation which, after substituting $e_{11} = Re_{22}$ from (1.17), resolves itself into a sextic in e_{22}, the coefficients of which are functions of the inertia and force coefficients $(a + \gamma)$, b and c. Theoretically, for an assumed value of the frequency parameter ω_m the b and c coefficients, and hence the coefficients of the sextic, could be evaluated and the sextic solved directly for e_{22} and hence for the speed from (1.17). The assumed value of ω_m is then checked from (1.29) and if different the process repeated until reasonable agreement is obtained. In practice this method is unsuitable because of the sextic solution.

The direct solution of the sextic can of course be avoided by adopting an indirect solution of equations (1.29). For a given speed, which with an assumed value of ω_m determines the values of the p coefficients, the second of equations (1.29) can be solved as a quadratic for ω_m^2 and its value substituted in the left-hand side of the first of equations (1.29). Repeating for a range of speeds, the speed for which the left-hand side of the first of equations (1.29) becomes zero can be found by interpolation.

In the second method, which in contrast to the first might be termed the "indirect non-iterative", (1.27) is expanded as a function of the stiffness coefficients e_{11} and e_{22}, giving

$$|\delta| + \Delta_{11}\, e_{11} + \Delta_{22}\, e_{22} + \delta_{33}\, e_{11}\, e_{22} = 0 \qquad (1.30)$$

where $|\delta|$ is the determinant of (1.27) with e_{11} and e_{22} omitted, and Δ_{rs} is the minor in $|\delta|$ of δ_{rs}.

Equating real and imaginary parts of (1.30) to zero then gives the two real equations

$$\left.\begin{aligned} R_o + R_1\, e_{11} + R_2\, e_{22} + R_3\, e_{11}\, e_{22} &= 0 \\ S_o + S_1\, e_{11} + S_2\, e_{22} + S_3\, e_{11}\, e_{22} &= 0 \end{aligned}\right\} \qquad (1.31)$$

where the R and S coefficients are functions of the original a, γ, b and c coefficients and the frequency parameter ω_m. For a given value of ω_m the R and S coefficients can be evaluated and equations (1.31) solved for e_{11} and e_{22}, and hence for the stiffness ratio $R = \frac{e_{11}}{e_{22}}$. This process involves only the solution of a quadratic. The procedure is therefore to obtain values of e_{11}, e_{22}, and R for a range of values of ω_m and to plot either e_{11} or e_{22} against R. For the actual value of R the corresponding speed and frequency are then obtained directly from the curve.

The above general forms of solution are amplified in greater detail in Appendix II.

Simplified Aileron Mode

For the purpose of a typical calculation some simplification is effected if instead of assuming the aileron torsionally rigid the aileron angle is assumed constant along the span and related directly to the co-ordinate q_3 by

$$\xi = \frac{\ell}{c_m} \cdot q_3 \qquad (1.32)$$

Comparing this with (1.6) for the torsionally rigid aileron, it is seen that the simplification involves the deletion of all terms containing the factor (1 - F) in the evaluation of the coefficients. In particular, the compound derivatives in the integrals for the aerodynamic coefficients become the basic derivatives; for example, ℓ_α' becomes ℓ_α.

4 Typical Binary Analysis. Wing Flexure and Torsion

The analysis for this case is obtained directly from that of Section 3 by simply omitting all those effects appropriate to the aileron motion. This means the omission of the coefficients of order 'r3' or '3s' and the deletion of all terms containing the factor (1 - F) from the remaining coefficients.

The equations of motion are now two and may be written down directly from (1.26) as

$$\left.\begin{aligned} (\delta_{11} + e_{11})\, q_1 + \delta_{12}\, q_2 &= 0 \\ \delta_{21}\, q_1 + (\delta_{22} + e_{22})\, q_2 &= 0 \end{aligned}\right\} \tag{2.1}$$

the corresponding determinantal equation being

$$\begin{vmatrix} \delta_{11} + e_{11}, & \delta_{12} \\ \delta_{21}, & \delta_{22} + e_{22} \end{vmatrix} = 0 \tag{2.2}$$

In this case the "direct iterative" method of solution is comparatively simple and is the one usually adopted for an 'ad hoc' determination of critical speed. Expanding (2.2) gives

$$p_o\, \lambda^4 + p_1\, \lambda^3 + p_2\, \lambda^2 + p_3\, \lambda + p_4 = 0 \tag{2.3}$$

which in turn gives rise to the two subsidiary equations

$$\left.\begin{aligned} p_o\, \omega_m^{\,4} - p_2\, \omega_m^{\,2} + p_4 &= 0 \\ - p_1\, \omega_m^{\,2} + p_3 &= 0 \end{aligned}\right\} \tag{2.4}$$

Eliminating ω_m from (2.4) gives finally

$$p_1\, p_2\, p_3 - p_o\, p_3^{\,2} - p_1^{\,2}\, p_4 = 0$$

which may be written alternatively as the test determinant

$$\begin{vmatrix} p_1 & p_o & 0 \\ p_3 & p_2 & p_1 \\ 0 & p_4 & p_3 \end{vmatrix} = 0 \tag{2.5}$$

Coefficients p_o to p_4 are as before functions of the inertia, stiffness, and force coefficients. Substituting $e_{11} = Re_{22}$, p_2 and p_3 become linear functions of e_{22}, p_4 a quadratic in e_{22}. p_o and p_1 are functions of inertia and force coefficients only. On expansion (2.5) then becomes a quadratic in e_{22}, instead of a sextic as in the case of the ternary.

For an assumed value of ω_m the coefficients of (2.5) are calculated and the equation solved for e_{22}. The assumed value of ω_m is then checked from $\omega_m^{\,2} = \dfrac{p_3}{p_1}$, the second of equations (2.4), and the process repeated until reasonable agreement is obtained. Finally the speed is obtained from (1.17) as

$$V = \frac{1}{c_m} \sqrt{\frac{m_\theta}{\rho \ell e_{22}}} \qquad (2.6)$$

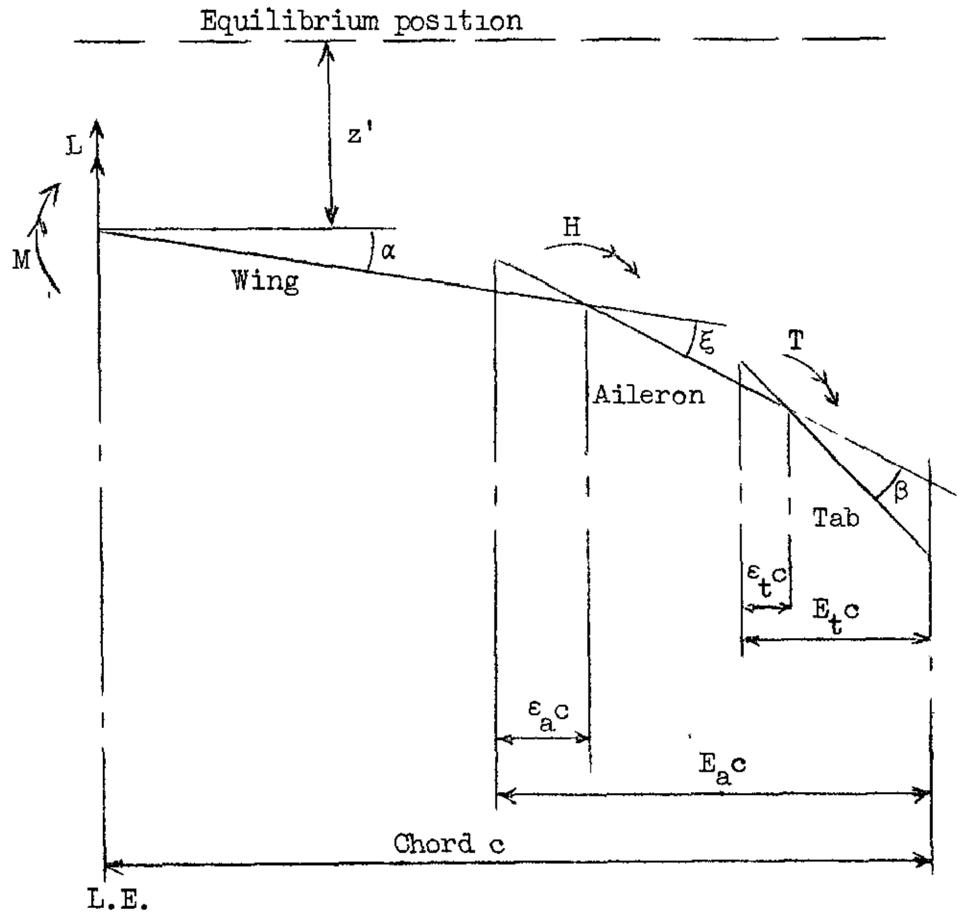
The calculation is fairly insensitive to the value taken for ω_m, so that iteration is quite often unnecessary. As an indication, if the initial value taken for ω_m is 1.0 the calculation need only be repeated if the check value is less than 0.6.

APPENDIX I

Aerodynamic Derivatives

(Two-dimensional, incompressible flow)

The wing motion and forces are referred, in the first instance, to the leading-edge as reference point. In the accompanying sketch forces and moments are represented by double-headed arrows.



If ω is the frequency parameter the forces are expressed as

$$\frac{L}{\rho c v^2} = (-\omega^2 \ell_{\ddot{z}} + i\omega \ell_{\dot{z}} + \ell_z)\frac{z'}{c} + (-\omega^2 \ell_{\ddot{\alpha}} + i\omega \ell_{\dot{\alpha}} + \ell_\alpha)\,\alpha$$

$$+ (-\omega^2 \ell_{\ddot{\xi}} + i\omega \ell_{\dot{\xi}} + \ell_\xi)\xi + (-\omega^2 \ell_{\ddot{\beta}} + i\omega \ell_{\dot{\beta}} + \ell_\beta)\,\beta$$

$$\frac{M}{\rho c^2 v^2} = (-\omega^2 m_{\ddot z} + i\omega m_{\dot z} + m_z)\frac{z'}{c} + (-\omega^2 m_{\ddot\alpha} + i\omega m_{\dot\alpha} + m_\alpha)\alpha$$

$$+ (-\omega^2 m_{\ddot\xi} + i\omega m_{\dot\xi} + m_\xi)\xi + (-\omega^2 m_{\ddot\beta} + i\omega m_{\dot\beta} + m_\beta)\beta$$

$$\frac{H}{\rho c^2 v^2} = (-\omega^2 h_{\ddot z} + i\omega h_{\dot z} + h_z)\frac{z'}{c} + (-\omega^2 h_{\ddot\alpha} + i\omega h_{\dot\alpha} + h_\alpha)\alpha$$

$$+ (-\omega^2 h_{\ddot\xi} + i\omega h_{\dot\xi} + h_\xi)\xi + (-\omega^2 h_{\ddot\beta} + i\omega h_{\dot\beta} + h_\beta)\beta$$

$$\frac{T}{\rho c^2 v^2} = (-\omega^2 t_{\ddot z} + i\omega t_{\dot z} + t_z)\frac{z'}{c} + (-\omega^2 t_{\ddot\alpha} + i\omega t_{\dot\alpha} + t_\alpha)\alpha$$

$$+ (-\omega^2 t_{\ddot\xi} + i\omega t_{\dot\xi} + t_\xi)\xi + (-\omega^2 t_{\ddot\beta} + i\omega t_{\dot\beta} + t_\beta)\beta$$

The coefficients are then

$\ell_{\ddot z} = \frac{1}{4}\pi$

$\ell_{\dot z} = \pi \mathbf{A}$

$\ell_z = \pi\omega B$

$\ell_{\ddot\alpha} = \frac{1}{8}\pi$

$\ell_{\dot\alpha} = \pi/4\ (1 + 3A - 4B/\omega)$

$\ell_\alpha = \frac{1}{4}\pi(4A + 3\omega B)$

$\ell_{\ddot\xi} = 1/16\ (\Phi_4 - 4\ \varepsilon_a\ \Phi_3)$

$\ell_{\dot\xi} = \frac{1}{4}\mathbf{A}(\Phi_2 - 4\ \varepsilon_a\ \Phi_1) - B\ \Phi_1/\omega + \frac{1}{4}\ \Phi_3$

$\ell_\xi = \mathbf{A}\ \Phi_1 + \frac{1}{4}\omega B\ (\Phi_2 - 4\ \varepsilon_a\ \Phi_1)$

$\ell_{\ddot\beta} = 1/16\ (\psi_4 - 4\ \varepsilon_t\ \psi_3)$

$\ell_{\dot\beta} = \frac{1}{4}\mathbf{A}(\psi_2 - 4\ \varepsilon_t\ \psi_1) - B\ \psi_1/\omega + \frac{1}{4}\ \psi_3$

$\ell_\beta = A\ \psi_1 + \frac{1}{4}\omega B\ (\psi_2 - 4\ \varepsilon_t\ \psi_1)$

$-m_{\ddot z} = \frac{1}{8}\pi$

$-m_{\dot z} = \frac{1}{4}\pi\mathbf{A}$

$-m_z = \frac{1}{4}\pi\omega B$

$$-m_{\ddot{\alpha}} = \frac{9}{128}\pi$$

$$-m_{\dot{\alpha}} = {}^1/16\ \pi(3 + 3A - 4B/\omega)$$

$$-m_{\alpha} = {}^1/16\ \pi(4A + 3\omega B)$$

$$-m_{\ddot{\xi}} = {}^1/64\ (\Phi_4 + \Phi_7) - {}^1/32\ \varepsilon_a\ (2\Phi_3 + \Phi_6)$$

$$-m_{\dot{\xi}} = {}^1/16\ (\Phi_3 + \Phi_6 - 4\ \varepsilon_a\ \Phi_5) + {}^1/16\ A(\Phi_2 - 4\ \varepsilon_a\ \Phi_1) - \tfrac{1}{4} B\ \Phi_1/\omega$$

$$-m_{\xi} = \tfrac{1}{4}\Phi_5 + \tfrac{1}{4} A\ \Phi_1 + {}^1/16\ \omega B\ (\Phi_2 - 4\ \varepsilon_a\ \Phi_1)$$

$$-m_{\ddot{\beta}} = {}^1/64\ (\psi_4 + \psi_7) - {}^1/32\ \varepsilon_t\ (2\ \psi_3 + \psi_6)$$

$$-m_{\dot{\beta}} = {}^1/16\ (\psi_3 + \psi_6 - 4\ \varepsilon_t\ \psi_5) + {}^1/16\ A\ (\psi_2 - 4\ \varepsilon_t\ \psi_1) - \tfrac{1}{4} B\ \psi_1/\omega$$

$$-m_{\beta} = \tfrac{1}{4}\psi_5 + \tfrac{1}{4} A\ \psi_1 + {}^1/16\ \omega B\ (\psi_2 - 4\ \varepsilon_t\ \psi_1)$$

$$-h_{\ddot{z}} = {}^1/16\ \Phi_4 - \tfrac{1}{4}\ \varepsilon_a\ \Phi_3$$

$$-h_{\dot{z}} = \tfrac{1}{4} A\ (\Phi_8 - 4\ \varepsilon_a\ \Phi_{31})$$

$$-h_{z} = \tfrac{1}{4}\omega B\ (\Phi_8 - 4\ \varepsilon_a\ \Phi_{31})$$

$$-h_{\ddot{\alpha}} = {}^1/64\ (\Phi_4 + \Phi_7) - {}^1/32\ \varepsilon_a\ (2\Phi_3 + \Phi_6)$$

$$-h_{\dot{\alpha}} = {}^1/16\ (\Phi_9 - 4\ \varepsilon_a\ \Phi_{32}) + {}^1/16\ (3A - 4B/\omega)\ (\Phi_8 - 4\ \varepsilon_a\ \Phi_{31})$$

$$-h_{\alpha} = {}^1/16\ (4A + 3\omega B)\ (\Phi_8 - 4\ \varepsilon_a\ \Phi_{31})$$

$$-h_{\ddot{\xi}} = \frac{1}{64\pi}(\Phi_{12} - 8\ \varepsilon_a\ \Phi_{37} + 16\ \varepsilon_a^{\ 2}\ \Phi_{17})$$

$$-h_{\dot{\xi}} = \frac{1}{16\pi}[\Phi_{11} - 4\ \varepsilon_a\ (\Phi_{10} + \Phi_{36}) + 8\ \varepsilon_a^{\ 2}\ \Phi_{35} + A\ (\Phi_2 - 4\ \varepsilon_a\ \Phi_1)(\Phi_8 - 4\ \varepsilon_a\ \Phi_{31}) - 4B\ \Phi_1(\Phi_8 - 4\ \varepsilon_a\ \Phi_{31})/\omega]$$

$$-h_{\xi} = \frac{1}{16\pi}[4\Phi_{10} - 8\ \varepsilon_a\ \Phi_{35} + 4A\ \Phi_1\ (\Phi_8 - 4\ \varepsilon_a\ \Phi_{31}) + \omega B(\Phi_2 - 4\ \varepsilon_a\ \Phi_1)(\Phi_8 - 4\ \varepsilon_a\ \Phi_{31})]$$

$$-h_{\ddot{\beta}} = \frac{1}{16\pi}[\chi_{10} - 2(\varepsilon_a\ \chi_5 + \varepsilon_t\ \chi_{18}) + 4\ \varepsilon_a\ \varepsilon_t\ \chi_{14}]$$

$$-h_{\dot{\beta}} = \frac{1}{8\pi}[\chi_9 - 2(\varepsilon_a\ \chi_4 + \varepsilon_t\ \chi_8) + 4\ \varepsilon_a\ \varepsilon_t\ \chi_3 + 2A\{\chi_7 - 2(\varepsilon_a\ \chi_2 + \varepsilon_t\ \chi_6) + 4\ \varepsilon_a\ \varepsilon_t\ \chi_1\} - 4B(\chi_6 - 2\ \varepsilon_a\ \chi_1)/\omega]$$

$$-h_\beta = \frac{1}{4\pi}\left[\chi_8 - 2\,\varepsilon_a\,\chi_3 + 2A(\chi_6 - 2\,\varepsilon_a\,\chi_1)\right.$$

$$\left. + \omega B\,\{\,\chi_7 - 2(\varepsilon_a\,\chi_2 + \varepsilon_t\,\chi_6) + 4\,\varepsilon_a\,\varepsilon_t\,\chi_1\}\,\right]$$

$$-t_{\ddot z} = 1/16\;\psi_4 - \tfrac{1}{4}\,\varepsilon_t\,\psi_3$$

$$-t_{\dot z} = \tfrac{1}{4}\,A\,(\psi_8 - 4\,\varepsilon_t\,\psi_{31})$$

$$-t_z = \tfrac{1}{4}\,\omega B\,(\psi_8 - 4\,\varepsilon_t\,\psi_{31})$$

$$-t_{\ddot\alpha} = {}^1/64\,(\psi_4 + \psi_7) - {}^1/32\;\varepsilon_t\,(2\psi_3 + \psi_6)$$

$$-t_{\dot\alpha} = {}^1/16\,(\psi_9 - 4\,\varepsilon_t\,\psi_{32}) + {}^1/16\,(3A - 4B/\omega)\,(\psi_8 - 4\,\varepsilon_t\,\psi_{31})$$

$$-t_\alpha = {}^1/16\,(4A + 3\,\omega B)\,(\psi_8 - 4\,\varepsilon_t\,\psi_{31})$$

$$-t_{\ddot\xi} = \frac{1}{16\pi}\left[\,X_{10} - 2(\varepsilon_a\,X_{18} + \varepsilon_t\,X_5) + 4\,\varepsilon_a\,\varepsilon_t\,X_{14}\,\right]$$

$$-t_{\dot\xi} = \frac{1}{8\pi}\left[\,X_9 - 2(\varepsilon_a\,X_8 + \varepsilon_t\,X_4) + 4\,\varepsilon_a\,\varepsilon_t\,X_3\right.$$

$$+ 2A\,\{X_7 - 2(\varepsilon_a\,X_6 + \varepsilon_t\,X_2) + 4\,\varepsilon_a\,\varepsilon_t\,X_1\}$$

$$\left. - 4B\,(X_6 - 2\,\varepsilon_t\,X_1)/\omega\,\right]$$

$$-t_\xi = \frac{1}{4\pi}\left[\,X_8 - 2\,\varepsilon_a\,X_3 + 2A\,(X_6 - 2\,\varepsilon_t\,X_1) + \omega B\,\{X_7 - 2(\varepsilon_t\,X_2 + \varepsilon_a\,X_6)\right.$$

$$\left. + 4\,\varepsilon_a\,\varepsilon_t\,X_1\}\,\right]$$

$$-t_{\ddot\beta} = \frac{1}{64\pi}\,(\psi_{12} - 8\,\varepsilon_t\,\psi_{37} + 16\,\varepsilon_t^{\,2}\,\psi_{17})$$

$$-t_{\dot\beta} = \frac{1}{16\pi}\left[\,\psi_{11} - 4\,\varepsilon_t\,(\psi_{10} + \psi_{36}) + 8\,\varepsilon_t^{\,2}\,\psi_{35}\right.$$

$$\left. + A(\psi_2 - 4\,\varepsilon_t\,\psi_1)(\psi_8 - 4\,\varepsilon_t\,\psi_{31}) - 4B\,\psi_1(\psi_8 - 4\,\varepsilon_t\,\psi_{31})/\omega\,\right]$$

$$-t_\beta = \frac{1}{16\pi}\left[\,4\psi_{10} - 8\,\varepsilon_t\,\psi_{35} + 4A\,\psi_1\,(\psi_8 - 4\,\varepsilon_t\,\psi_{31})\right.$$

$$\left. + \omega B(\psi_2 - 4\,\varepsilon_t\,\psi_1)(\psi_8 - 4\,\varepsilon_t\,\psi_{31})\,\right]$$

Transformation to alternative reference axis

Derivatives for an alternative reference axis may be obtained from those for the leading edge as reference axis by simple transformation formulae. If the alternative reference axis is situated a distance hc behind the leading edge and its displacement is z, then the displacement at the leading edge is given by

$$z' = z - hc\alpha$$

and the moment about the alternative reference axis is

$$M(h) = M + hcL$$

Making these substitutions, the forces referred to the reference axis at hc may be written, using the complex derivative notation employed in Section 3, as

$$\frac{L(h)}{\rho c V^2} = L_z(h)\,\frac{z}{c} + L_\alpha(h)\,\alpha + L_\xi(h)\,\xi + L_\beta(h)\,\beta$$

with similar expressions for $M(h)$, $H(h)$ and $T(h)$, where

$$L_z(h) = L_z, \quad L_\alpha(h) = L_\alpha - hL_z, \quad L_\xi(h) = L_\xi, \quad L_\beta(h) = L_\beta$$

$$M_z(h) = M_z + hL_z, \quad M_\alpha(h) = M_\alpha - hM_z + hL_\alpha - h^2L_z$$

$$M_\xi(h) = M_\xi + hL_\xi, \quad M_\beta(h) = M_\beta + hL_\beta$$

$$H_z(h) = H_z, \quad H_\alpha(h) = H_\alpha - hH_z, \quad H_\xi(h) = H_\xi, \quad H_\beta(h) = H_\beta$$

$$T_z(h) = T_z, \quad T_\alpha(h) = T_\alpha - hT_z, \quad T_\xi(h) = T_\xi, \quad T_\beta(h) = T_\beta$$

Derivatives without the suffix (h) are those for the leading edge as reference axis. Relationships between the basic derivatives are the same as those above between the complex derivatives, e.g.

$$\ell_\alpha(h) = \ell_\alpha - h\ell_z$$

$$m_{\dot\alpha}(h) = m_{\dot\alpha} - hm_{\dot z} + h\ell_{\dot\alpha} - h^2\ell_{\dot z}$$

Table 1.

Values of A and B

ω	A	B
0	1.0000000	0.0000000
0.02	0.9824216	0.0456521
0.04	0.9637253	0.0752079
0.06	0.9450111	0.0979135
0.08	0.9267018	0.1160013
0.10	0.9090087	0.1306443
0.12	0.8920397	0.1425944
0.16	0.8604318	0.1604021
0.20	0.8319241	0.1723022
0.24	0.8063273	0.1800727
0.28	0.7833715	0.1848904
0.32	0.7627719	0.1875659
0.36	0.7442570	0.1886727
0.40	0.7275799	0.1886242
0.44	0.7125211	0.1877232
0.48	0.6988879	0.1861940
0.52	0.6865125	0.1842043
0.56	0.6752492	0.1818807
0.60	0.6649711	0.1793191
0.64	0.6555686	0.1765929
0.68	0.6469460	0.1737580
0.72	0.6390200	0.1708575
0.76	0.6317179	0.1679244
0.80	0.6249763	0.1649840
0.84	0.6187392	0.1620556
0.88	0.6129575	0.1591543
0.92	0.6075879	0.1562909
0.96	0.6025921	0.1534740
1.00	0.5979361	0.1507095
1.04	0.5935896	0.1480019
1.08	0.5895258	0.1453541
1.12	0.5857205	0.1427682
1.16	0.5821522	0.1402450
1.20	0.5788016	0.1377852
1.24	0.5756512	0.1353835

ω	A	B
1.28	0.5726853	0.1330545
1.32	0.5698898	0.1307822
1.36	0.5672518	0.1285708
1.40	0.5647596	0.1264189
1.44	0.5624026	0.1243252
1.48	0.5601712	0.1222882
1.52	0.5580567	0.1203065
1.56	0.5560509	0.1183784
1.60	0.5541466	0.1165024
1.64	0.5523369	0.1146768
1.72	0.5489774	0.1111714
1.80	0.5459286	0.1078496
1.88	0.5431533	0.1046996
1.96	0.5406197	0.1017105
2.00	0.5394349	0.1002729
2.20	0.5342148	0.0936062
2.40	0.5299560	0.0877090
2.60	0.5264367	0.0824643
2.80	0.5234957	0.0777759
3.00	0.5210132	0.0735641
3.20	0.5188992	0.0697629
3.40	0.5170845	0.0663173_5
3.60	0.5155155	0.0631816_5
3.80	0.5141501	0.0603171_5
4.00	0.5129548	0.0576913
4.20	0.5119026	0.0552762
4.40	0.5109717	0.0530482_5
4.60	0.5101443	0.0509871
4.80	0.5094058	0.0490750
5.00	0.5087440	0.0472969
6.00	0.5062799	0.0400039
8.00	0.5036709	0.0304961
10.00	0.5023973	0.0245986
20.00	0.5006178	0.0124467
∞	0.5000000	0.0000000

Table 2.

Values of Φ

E_a	$-\cos\varphi$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8
0.60	-0.2	2.75195	4.63657	1.96811	2.04138	0.78384	5.19037	3.08815	0.70034
0.55	-0.1	2.66595	4.09463	1.77046	1.66748	0.89549	4.85431	2.58554	0.55371
0.50	0.0	2.57080	3.57080	1.57080	1.33333	1.00000	4.47493	2.11873	0.42920
0.45	0.1	2.46562	3.06698	1.37113	1.03916	1.09449	4.05564	1.69189	0.32472
0.40	0.2	2.34923	2.58530	1.17348	0.78475	1.17576	3.60110	1.30878	0.23834
0.35	0.3	2.22004	2.12814	0.97992	0.56949	1.24012	3.11729	0.97265	0.16829
0.30	0.4	2.07579	1.69828	0.79267	0.39236	1.28312	2.61184	0.68605	0.11293
0.25	0.5	1.91322	1.29904	0.61418	0.25184	1.29904	2.09440	0.45068	0.07067
0.20	0.6	1.72730	0.93454	0.44730	0.14591	1.28000	1.57726	0.26716	0.03995
0.15	0.7	1.50954	0.61023	0.29550	0.07192	1.21404	1.07661	0.13469	0.01923
0.10	0.8	1.24350	0.33390	0.16350	0.02640	1.08000	0.61500	0.05055	0.00690
0.05	0.9	0.88692	0.11866	0.05873	0.00472	0.82819	0.22788	0.00924	0.00121
0.00	1.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.10	0.80	1.24350	0.33390	0.16350	0.02640	1.08000	0.61500	0.05055	0.00690
0.09	0.82	1.18175	0.28538	0.14005	0.02033	1.04170	0.53010	0.03910	0.00529
0.08	0.84	1.11610	0.23941	0.11774	0.01518	0.99836	0.44846	0.02932	0.00393
0.07	0.86	1.04582	0.19616	0.09667	0.01089	0.94915	0.37052	0.02114	0.00281
0.06	0.88	0.96991	0.15582	0.07696	0.00743	0.89295	0.29679	0.01447	0.00191
0.05	0.90	0.88692	0.11866	0.05873	0.00472	0.82819	0.22788	0.00924	0.00121
0.04	0.92	0.79463	0.08499	0.04215	0.00271	0.75248	0.16457	0.00532	0.00069
0.03	0.94	0.68934	0.05526	0.02746	0.00132	0.66188	0.10787	0.00261	0.00033
0.02	0.96	0.56379	0.03011	0.01499	0.00048	0.54880	0.05926	0.00095	0.00012
0.01	0.98	0.39933	0.01066	0.00532	0.00009	0.39402	0.02114	0.00017	0.00002
0.00	1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 2 (Cont'd)

Values of Φ

E_a	-cos φ	Φ_9	Φ_{10}	Φ_{11}	Φ_{12}	Φ_{13}	Φ_{14}	Φ_{15}	Φ_{16}	Φ_{17}
0.60	-0.2	3.38243	0.62108	9.12529	6.54742	0.81650	1.95959	-1.14310	5.39270	4.79507
0.55	-0.1	2.78125	0.60533	7.24939	4.67963	0.90453	1.98997	-1.08544	5.30518	4.11464
0.50	0.0	2.23746	0.57080	5.60899	3.23370	1.00000	2.00000	-1.00000	5.14159	3.46740
0.45	0.1	1.75360	0.52058	4.20523	2.14543	1.10554	1.98997	-0.88443	4.90651	2.86010
0.40	0.2	1.33116	0.45812	3.03379	1.35379	1.22474	1.95959	-0.73485	4.60354	2.29865
0.35	0.3	0.97069	0.38712	2.08541	0.80178	1.36277	1.90788	-0.54511	4.23557	1.78835
0.30	0.4	0.67178	0.31150	1.34618	0.43709	1.52753	1.83303	-0.30551	3.80499	1.33393
0.25	0.5	0.43301	0.23535	0.79785	0.21281	1.73205	1.73205	0.00000	3.31380	0.93972
0.20	0.6	0.25187	0.16294	0.41802	0.08797	2.00000	1.60000	0.40000	2.76367	0.60967
0.15	0.7	0.12461	0.09865	0.18032	0.02309	2.38048	1.42829	0.95219	2.15606	0.34742
0.10	0.8	0.04590	0.04698	0.05459	0.00560	3.00000	1.20000	1.80000	1.49220	0.15633
0.05	0.9	0.00823	0.01254	0.00697	0.00035	4.35890	0.87178	3.48712	0.77320	0.03955
0.00	1.0	0.00000	0.00000	0.00000	0.00000	∞	0.00000	∞	0.00000	0.00000
0.10	0.80	0.04590	0.04698	0.05459	0.00560	3.00000	1.20000	1.80000	1.49220	0.15633
0.09	0.82	0.03537	0.03857	0.03997	0.00368	3.17980	1.14473	2.03507	1.35278	0.12694
0.08	0.84	0.02643	0.03088	0.02819	0.00230	3.39117	1.08517	2.30599	1.21116	0.10053
0.07	0.86	0.01898	0.02395	0.01896	0.00135	3.64496	1.02059	2.62437	1.06735	0.07715
0.06	0.88	0.01295	0.01782	0.01199	0.00073	3.95811	0.94995	3.00817	0.92136	0.05682
0.05	0.90	0.00823	0.01254	0.00697	0.00035	4.35890	0.87178	3.48712	0.77320	0.03955
0.04	0.92	0.00473	0.00812	0.00358	0.00014	4.89898	0.78384	4.11514	0.62286	0.02537
0.03	0.94	0.00231	0.00463	0.00152	0.00005	5.68624	0.68235	5.00389	0.47037	0.01430
0.02	0.96	0.00084	0.00208	0.00045	0.00001	7.00000	0.56000	6.44000	0.31573	0.00637
0.01	0.98	0.00015	0.00053	0.00006	0.00000	9.94987	0.39800	9.55188	0.15893	0.00160
0.00	1.00	0.00000	0.00000	0.00000	0.00000	∞	0.00000	∞	0.00000	0.00000

Table 2 (Cont'd)

Values of Φ

E_a	$-\cos\varphi$	$-\Phi_{18}$	Φ_{19}	Φ_{20}	Φ_{21}	Φ_{31}	Φ_{32}	Φ_{35}	Φ_{36}	Φ_{37}
0.60	-0.2	1.86574	1.92835	1.17576	-0.31836	0.79236	3.14387	1.92000	8.03069	5.25182
0.55	-0.1	1.72373	1.76159	1.09449	-0.17990	0.67598	2.86495	1.98000	7.03248	4.11485
0.50	0.0	1.57080	1.57080	1.00000	0.00000	0.57080	2.57080	2.00000	6.03820	3.14159
0.45	0.1	1.41067	1.36426	0.89549	0.22010	0.47564	2.26662	1.98000	5.06803	2.32523
0.40	0.2	1.24633	1.14977	0.78384	0.48164	0.38964	1.95732	1.92000	4.14007	1.65674
0.35	0.3	1.08016	0.93479	0.66776	0.78862	0.31216	1.64768	1.82000	3.27080	1.12516
0.30	0.4	0.91417	0.72650	0.54991	1.14871	0.24276	1.34258	1.68000	2.47543	0.71785
0.25	0.5	0.75000	0.53190	0.43301	1.57536	0.18117	1.04720	1.50000	1.76817	0.42063
0.20	0.6	0.58908	0.35784	0.32000	2.09257	0.12730	0.76730	1.28000	1.16241	0.21794
0.15	0.7	0.43263	0.21103	0.21424	2.74669	0.08126	0.50974	1.02000	0.67083	0.09300
0.10	0.8	0.28170	0.09810	0.12000	3.64330	0.04350	0.28350	0.72000	0.30555	0.02786
0.05	0.9	0.13722	0.02560	0.04359	5.12146	0.01514	0.10231	0.38000	0.07821	0.00352
0.00	1.0	0.00000	0.00000	0.00000	∞	0.00000	0.00000	0.00000	0.00000	0.00000
0.10	0.80	0.28170	0.09810	0.12000	3.64330	0.04350	0.28350	0.72000	0.30555	0.02786
0.09	0.82	0.25226	0.08016	0.10303	3.87192	0.03702	0.24307	0.65520	0.24869	0.02035
0.08	0.84	0.22308	0.06388	0.08681	4.12563	0.03093	0.20455	0.58880	0.19743	0.01433
0.07	0.86	0.19418	0.04933	0.07144	4.41107	0.02523	0.16812	0.52080	0.15187	0.00962
0.06	0.88	0.16556	0.03655	0.05700	4.73798	0.01996	0.13395	0.45120	0.11210	0.00607
0.05	0.90	0.13722	0.02560	0.04359	5.12146	0.01514	0.10231	0.38000	0.07821	0.00352
0.04	0.92	0.10917	0.01652	0.03135	5.58681	0.01080	0.07350	0.30720	0.05028	0.00181
0.03	0.94	0.08142	0.00937	0.02047	6.18145	0.00699	0.04793	0.23280	0.02841	0.00076
0.02	0.96	0.05397	0.00420	0.01120	7.01186	0.00379	0.02619	0.15680	0.01269	0.00023
0.01	0.98	0.02683	0.00106	0.00398	8.41785	0.00134	0.00930	0.07920	0.00319	0.00003
0.00	1.00	0.00000	0.00000	0.00000	∞	0.00000	0.00000	0.00000	0.00000	0.00000

Note The values of the functions ψ are the same as the corresponding functions Φ except that they refer to the tab, and the value of E_t should be substituted for E_a.

Table 3. Function X_3

$-\cos\varphi$	$-\cos\varphi$	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	0.05895	0.07373	0.09096	0.11122	0.13534	0.16463	0.20123	0.24906	0.31622	0.42466	0.72000
0.82	0.09	0.04997	0.06247	0.07704	0.09413	0.11445	0.13906	0.16971	0.20951	0.26484	0.35199	0.54535
0.84	0.08	0.04158	0.05196	0.06405	0.07821	0.09501	0.11532	0.14053	0.17309	0.21792	0.28712	0.42762
0.86	0.07	0.03379	0.04222	0.05201	0.06347	0.07705	0.09343	0.11369	0.13973	0.17529	0.22922	0.33246
0.88	0.06	0.02663	0.03326	0.04095	0.04995	0.06060	0.07340	0.08920	0.10942	0.13681	0.17774	0.25274
0.90	0.05	0.02012	0.02512	0.03092	0.03769	0.04569	0.05530	0.06711	0.08217	0.10243	0.13231	0.18522
0.92	0.04	0.01430	0.01785	0.02196	0.02675	0.03241	0.03919	0.04751	0.05806	0.07218	0.09276	0.12820
0.94	0.03	0.00923	0.01151	0.01416	0.01724	0.02087	0.02522	0.03054	0.03726	0.04620	0.05910	0.08080
0.96	0.02	0.00499	0.00622	0.00765	0.00931	0.01127	0.01360	0.01645	0.02004	0.02479	0.03158	0.04279
0.98	0.01	0.00175	0.00219	0.00269	0.00327	0.00395	0.00477	0.00576	0.00701	0.00865	0.01097	0.01475
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 4. Function λ_3

$-\cos\varphi$	$-\cos\varphi$	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	2.29256	2.31424	2.30904	2.27675	2.21617	2.12483	1.99840	1.82941	1.60378	1.28929	0.72000
0.82	0.09	2.19323	2.21550	2.21242	2.18385	2.12875	2.04494	1.92861	1.77321	1.56673	1.28301	0.82832
0.84	0.08	2.08492	2.10750	2.10630	2.08126	2.03148	1.95506	1.84863	1.70649	1.51836	1.26282	0.87459
0.86	0.07	1.96614	1.98873	1.98917	1.96747	1.92288	1.85373	1.75708	1.62798	1.45766	1.22847	0.89224
0.88	0.06	1.83488	1.85711	1.85894	1.84042	1.80091	1.73898	1.65208	1.53594	1.38311	1.17906	0.88720
0.90	0.05	1.68821	1.70970	1.71264	1.69713	1.66264	1.60795	1.53089	1.42780	1.29242	1.11284	0.86092
0.92	0.04	1.52170	1.54197	1.54572	1.53306	1.50359	1.45627	1.38929	1.29958	1.18196	1.02678	0.81240
0.94	0.03	1.32790	1.34635	1.35054	1.34062	1.31625	1.27662	1.22023	1.14460	1.04556	0.91549	0.73802
0.96	0.02	1.09238	1.10816	1.11235	1.10507	1.08610	1.05481	1.01004	0.94990	0.87121	0.76826	0.62921
0.98	0.01	0.77815	0.78981	0.79330	0.78873	0.77596	0.75456	0.72378	0.68234	0.62815	0.55748	0.46285
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5. Function χ_4

$-\cos\varphi$	$-\cos\varphi$	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	0.80503	0.78320	0.75809	0.72925	0.69607	0.65775	0.61317	0.56061	0.49725	0.41761	0.30555
0.82	0.09	0.69088	0.67234	0.65104	0.62658	0.59846	0.56603	0.52834	0.48400	0.43073	0.36422	0.27318
0.84	0.08	0.58193	0.56648	0.54874	0.52838	0.50499	0.47804	0.44676	0.41003	0.36604	0.31143	0.23807
0.86	0.07	0.47869	0.46612	0.45169	0.43513	0.41613	0.39425	0.36889	0.33917	0.30366	0.25982	0.20175
0.88	0.06	0.38175	0.37183	0.36045	0.34740	0.33243	0.31521	0.29527	0.27193	0.24413	0.20997	0.16523
0.90	0.05	0.29184	0.28433	0.27572	0.26586	0.25455	0.24155	0.22652	0.20896	0.18808	0.16253	0.12940
0.92	0.04	0.20984	0.20449	0.19837	0.19135	0.18332	0.17409	0.16343	0.15099	0.13624	0.11826	0.09513
0.94	0.03	0.13695	0.13350	0.12954	0.12501	0.11983	0.11388	0.10702	0.09902	0.08955	0.07805	0.06337
0.96	0.02	0.07490	0.07303	0.07089	0.06844	0.06564	0.06242	0.05872	0.05441	0.04931	0.04315	0.03532
0.98	0.01	0.02661	0.02595	0.02520	0.02434	0.02335	0.02222	0.02092	0.01941	0.01763	0.01548	0.01277
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 6. Function χ_4

$-\cos\varphi$	$-\cos\varphi$	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	0.76110	0.74676	0.73310	0.71112	0.68341	0.64934	0.60796	0.55770	0.49590	0.41718	0.30555
0.82	0.09	0.65277	0.64070	0.62925	0.61073	0.58734	0.55859	0.52369	0.48137	0.42947	0.36378	0.27315
0.84	0.08	0.54950	0.53953	0.53011	0.51478	0.49541	0.47159	0.44269	0.40770	0.36489	0.31100	0.23800
0.86	0.07	0.45175	0.44370	0.43614	0.42374	0.40807	0.38880	0.36542	0.33715	0.30264	0.25942	0.20167
0.88	0.06	0.36006	0.35377	0.34787	0.33816	0.32586	0.31074	0.29241	0.27025	0.24327	0.20961	0.16515
0.90	0.05	0.27510	0.27038	0.26598	0.25868	0.24943	0.23805	0.22426	0.20761	0.18738	0.16223	0.12932
0.92	0.04	0.19769	0.19436	0.19127	0.18611	0.17957	0.17151	0.16176	0.14999	0.13570	0.11802	0.09506
0.94	0.03	0.12895	0.12682	0.12485	0.12154	0.11733	0.11216	0.10589	0.09834	0.08918	0.07789	0.06331
0.96	0.02	0.07049	0.06934	0.06829	0.06651	0.06425	0.06146	0.05809	0.05402	0.04910	0.04305	0.03529
0.98	0.01	0.02503	0.02463	0.02426	0.02364	0.02285	0.02187	0.02069	0.01927	0.01755	0.01545	0.01276
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 7. Function X_5

-cos φ	-cos φ / E_a E_t	-0.2 / 0.60	-0.1 / 0.55	0.0 / 0.50	0.1 / 0.45	0.2 / 0.40	0.3 / 0.35	0.4 / 0.30	0.5 / 0.25	0.6 / 0.20	0.7 / 0.15	0.8 / 0.10
0.80	0.10	0.34977	0.30598	0.26380	0.22340	0.18498	0.14877	0.11504	0.08413	0.05648	0.03272	0.01393
0.82	0.09	0.30208	0.26453	0.22834	0.19366	0.16066	0.12954	0.10051	0.07387	0.04998	0.02935	0.01285
0.84	0.08	0.25606	0.22445	0.19397	0.16476	0.13694	0.11069	0.08618	0.06364	0.04339	0.02582	0.01162
0.86	0.07	0.21196	0.18598	0.16092	0.13688	0.11398	0.09235	0.07214	0.05353	0.03677	0.02217	0.01026
0.88	0.06	0.17011	0.14940	0.12942	0.11024	0.09197	0.07470	0.05854	0.04364	0.03019	0.01843	0.00877
0.90	0.05	0.13086	0.11504	0.09977	0.08511	0.07113	0.05790	0.04552	0.03409	0.02375	0.01468	0.00717
0.92	0.04	0.09468	0.08331	0.07233	0.06179	0.05173	0.04221	0.03329	0.02504	0.01756	0.01099	0.00550
0.94	0.03	0.06218	0.05476	0.04760	0.04072	0.03415	0.02793	0.02209	0.01669	0.01178	0.00745	0.00382
0.96	0.02	0.03422	0.03017	0.02625	0.02249	0.01889	0.01548	0.01228	0.00932	0.00662	0.00424	0.00222
0.98	0.01	0.01223	0.01079	0.00940	0.00806	0.00679	0.00557	0.00444	0.00338	0.00242	0.00156	0.00084
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 8. Function X_5

-cos φ	-cos φ / E_a E_t	-0.2 / 0.60	-0.1 / 0.55	0.0 / 0.50	0.1 / 0.45	0.2 / 0.40	0.3 / 0.35	0.4 / 0.30	0.5 / 0.25	0.6 / 0.20	0.7 / 0.15	0.8 / 0.10
0.80	0.10	0.03606	0.03486	0.03353	0.03205	0.03041	0.02858	0.02652	0.02418	0.02146	0.01819	0.01393
0.82	0.09	0.02779	0.02688	0.02586	0.02473	0.02347	0.02207	0.02050	0.01871	0.01664	0.01416	0.01097
0.84	0.08	0.02077	0.02009	0.01933	0.01849	0.01756	0.01652	0.01536	0.01404	0.01251	0.01068	0.00835
0.86	0.07	0.01492	0.01443	0.01389	0.01329	0.01263	0.01189	0.01106	0.01012	0.00904	0.00774	0.00611
0.88	0.06	0.01018	0.00985	0.00948	0.00908	0.00863	0.00813	0.00757	0.00693	0.00620	0.00533	0.00424
0.90	0.05	0.00647	0.00626	0.00603	0.00578	0.00549	0.00518	0.00482	0.00442	0.00396	0.00342	0.00274
0.92	0.04	0.00371	0.00360	0.00346	0.00332	0.00316	0.00298	0.00278	0.00255	0.00229	0.00198	0.00159
0.94	0.03	0.00181	0.00176	0.00169	0.00162	0.00154	0.00146	0.00136	0.00125	0.00112	0.00097	0.00079
0.96	0.02	0.00066	0.00064	0.00062	0.00059	0.00056	0.00053	0.00050	0.00046	0.00041	0.00036	0.00029
0.98	0.01	0.00012	0.00011	0.00011	0.00010	0.00010	0.00009	0.00009	0.00008	0.00007	0.00006	0.00005
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 9. Function χ_8

$-\cos\varphi$	$-\cos\varphi$	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	0.00462	0.00578	0.00712	0.00868	0.01055	0.01279	0.01557	0.01915	0.02406	0.03158	0.04698
0.82	0.09	0.00353	0.00441	0.00544	0.00663	0.00805	0.00976	0.01187	0.01457	0.01826	0.02383	0.03453
0.84	0.08	0.00262	0.00327	0.00403	0.00491	0.00596	0.00722	0.00877	0.01075	0.01344	0.01745	0.02485
0.86	0.07	0.00187	0.00233	0.00287	0.00350	0.00424	0.00513	0.00623	0.00763	0.00951	0.01230	0.01728
0.88	0.06	0.00126	0.00158	0.00194	0.00236	0.00287	0.00347	0.00420	0.00514	0.00640	0.00824	0.01145
0.90	0.05	0.00080	0.00099	0.00122	0.00149	0.00181	0.00218	0.00264	0.00323	0.00401	0.00515	0.00709
0.92	0.04	0.00045	0.00057	0.00070	0.00085	0.00103	0.00124	0.00150	0.00183	0.00227	0.00291	0.00397
0.94	0.03	0.00022	0.00027	0.00034	0.00041	0.00050	0.00060	0.00073	0.00089	0.00110	0.00140	0.00189
0.96	0.02	0.00008	0.00010	0.00012	0.00015	0.00018	0.00022	0.00026	0.00032	0.00039	0.00050	0.00067
0.98	0.01	0.00001	0.00002	0.00002	0.00003	0.00003	0.00004	0.00005	0.00006	0.00007	0.00009	0.00012
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 10. Function χ_8

$-\cos\varphi$	$-\cos\varphi$	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	2.00203	1.77146	1.54007	1.31056	1.08567	0.86834	0.66187	0.47008	0.29786	0.15222	0.04698
0.82	0.09	1.94395	1.72330	1.50169	1.28166	1.06580	0.85687	0.65790	0.47244	0.30494	0.16163	0.05360
0.84	0.08	1.87521	1.66539	1.45451	1.24493	1.03908	0.83952	0.64906	0.47098	0.30929	0.16952	0.06092
0.86	0.07	1.79412	1.59619	1.39711	1.19909	1.00438	0.81533	0.63455	0.46500	0.31032	0.17540	0.06806
0.88	0.06	1.69837	1.51360	1.32763	1.14249	0.96024	0.78305	0.61328	0.45361	0.30730	0.17867	0.07433
0.90	0.05	1.58474	1.41469	1.24342	1.07277	0.90462	0.74092	0.58378	0.43560	0.29929	0.17858	0.07907
0.92	0.04	1.44839	1.29506	1.14054	0.98647	0.83449	0.68634	0.54389	0.40924	0.28490	0.17409	0.08148
0.94	0.03	1.28135	1.14751	1.01255	0.87788	0.74491	0.61513	0.49014	0.37173	0.26200	0.16364	0.08045
0.96	0.02	1.06843	0.95830	0.84718	0.73621	0.62656	0.51940	0.41604	0.31791	0.22668	0.14447	0.07421
0.98	0.01	0.77131	0.69284	0.61362	0.53445	0.45615	0.37955	0.30555	0.23515	0.16950	0.11006	0.05880
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 11. Function X_9

-cos φ	-cos φ	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	0.06534	0.06362	0.06164	0.05938	0.05678	0.05379	0.05032	0.04624	0.04138	0.03535	0.02730
0.82	0.09	0.05039	0.04907	0.04756	0.04583	0.04384	0.04156	0.03891	0.03580	0.03210	0.02754	0.02150
0.84	0.08	0.03767	0.03669	0.03557	0.03429	0.03281	0.03112	0.02916	0.02687	0.02414	0.02078	0.01639
0.86	0.07	0.02707	0.02637	0.02558	0.02466	0.02361	0.02241	0.02101	0.01938	0.01744	0.01507	0.01199
0.88	0.06	0.01848	0.01801	0.01746	0.01685	0.01614	0.01532	0.01438	0.01328	0.01197	0.01038	0.00832
0.90	0.05	0.01175	0.01146	0.01111	0.01072	0.01028	0.00976	0.00917	0.00848	0.00765	0.00666	0.00537
0.92	0.04	0.00675	0.00658	0.00639	0.00616	0.00591	0.00562	0.00528	0.00488	0.00442	0.00385	0.00313
0.94	0.03	0.00330	0.00322	0.00312	0.00302	0.00289	0.00275	0.00259	0.00240	0.00217	0.00190	0.00155
0.96	0.02	0.00120	0.00117	0.00114	0.00110	0.00105	0.00100	0.00094	0.00088	0.00079	0.00070	0.00057
0.98	0.01	0.00021	0.00021	0.00020	0.00020	0.00019	0.00018	0.00017	0.00016	0.00014	0.00012	0.00010
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 12. Function X_9

-cos φ	-cos φ	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	0.64228	0.56670	0.49251	0.42026	0.35048	0.28379	0.22086	0.16249	0.10970	0.06387	0.02730
0.82	0.09	0.55512	0.49028	0.42662	0.36458	0.30463	0.24729	0.19312	0.14280	0.09716	0.05736	0.02521
0.84	0.08	0.47088	0.41629	0.36268	0.31040	0.25985	0.21147	0.16571	0.12313	0.08442	0.05052	0.02284
0.86	0.07	0.39008	0.34519	0.30109	0.25807	0.21645	0.17657	0.13882	0.10365	0.07160	0.04341	0.02019
0.88	0.06	0.31327	0.27749	0.24232	0.20799	0.17477	0.14291	0.11273	0.08456	0.05884	0.03613	0.01727
0.90	0.05	0.24115	0.21381	0.18693	0.16068	0.13526	0.11086	0.08773	0.06611	0.04632	0.02879	0.01413
0.92	0.04	0.17460	0.15495	0.13562	0.11674	0.09844	0.08088	0.06420	0.04859	0.03428	0.02157	0.01085
0.94	0.03	0.11474	0.10192	0.08931	0.07698	0.06503	0.05355	0.04263	0.03241	0.02302	0.01465	0.00755
0.96	0.02	0.06319	0.05618	0.04928	0.04254	0.03599	0.02970	0.02372	0.01811	0.01295	0.00833	0.00439
0.98	0.01	0.02260	0.02011	0.01766	0.01526	0.01294	0.01070	0.00857	0.00657	0.00473	0.00307	0.00166
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 13. Function $\chi_{10} = \chi_{10}$

-cos φ	-cos φ	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	0.02891	0.02536	0.02194	0.01866	0.01554	0.01259	0.00983	0.00729	0.00500	0.00302	0.00140
0.82	0.09	0.02239	0.01966	0.01702	0.01449	0.01208	0.00980	0.00767	0.00571	0.00394	0.00240	0.00113
0.84	0.08	0.01682	0.01477	0.01280	0.01091	0.00911	0.00740	0.00581	0.00434	0.00301	0.00184	0.00089
0.86	0.07	0.01214	0.01067	0.00926	0.00790	0.00660	0.00537	0.00422	0.00316	0.00220	0.00136	0.00067
0.88	0.06	0.00832	0.00732	0.00636	0.00543	0.00454	0.00370	0.00292	0.00219	0.00154	0.00096	0.00048
0.90	0.05	0.00532	0.00468	0.00407	0.00348	0.00291	0.00238	0.00188	0.00142	0.00100	0.00063	0.00032
0.92	0.04	0.00307	0.00270	0.00235	0.00201	0.00169	0.00138	0.00109	0.00083	0.00058	0.00037	0.00019
0.94	0.03	0.00151	0.00133	0.00116	0.00099	0.00083	0.00068	0.00054	0.00041	0.00029	0.00019	0.00010
0.96	0.02	0.00055	0.00049	0.00042	0.00036	0.00031	0.00025	0.00020	0.00015	0.00011	0.00007	0.00004
0.98	0.01	0.00010	0.00009	0.00008	0.00006	0.00005	0.00004	0.00004	0.00003	0.00002	0.00001	0.00001
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 14. Function χ_{12}

-cos φ	-cos φ	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	-0.13701	-0.15928	-0.18629	-0.22016	-0.26434	-0.32488	-0.41343	-0.55545	-0.81888	-1.46659	-∞
0.82	0.09	-0.11595	-0.13468	-0.15733	-0.18565	-0.22244	-0.27258	-0.34534	-0.46056	-0.66939	-0.15428	-3.64024
0.84	0.08	-0.09633	-0.11180	-0.13045	-0.15371	-0.18381	-0.22461	-0.28339	-0.37542	-0.53889	-0.90142	-2.33157
0.86	0.07	-0.07818	-0.09065	-0.10567	-0.12433	-0.14839	-0.18087	-0.22732	-0.29931	-0.42492	-0.69317	-1.60309
0.88	0.06	-0.06152	-0.07128	-0.08300	-0.09753	-0.11620	-0.14128	-0.17693	-0.23165	-0.32565	-0.52013	-1.11555
0.90	0.05	-0.04642	-0.05374	-0.06251	-0.07336	-0.08726	-0.10584	-0.13211	-0.17207	-0.23976	-0.37608	-0.76313
0.92	0.04	-0.03294	-0.03811	-0.04429	-0.05192	-0.06165	-0.07462	-0.09284	-0.12035	-0.16636	-0.25688	-0.49932
0.94	0.03	-0.02123	-0.02454	-0.02849	-0.03336	-0.03955	-0.04777	-0.05926	-0.07648	-0.10495	-0.15984	-0.30015
0.96	0.02	-0.01146	-0.01324	-0.01536	-0.01797	-0.02127	-0.02564	-0.03172	-0.04077	-0.05557	-0.08361	-0.15258
0.98	0.01	-0.00402	-0.00464	-0.00538	-0.00629	-0.00743	-0.00894	-0.01103	-0.01413	-0.01914	-0.02848	-0.05074
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 15. Function X_{12}

	-cos φ	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
-cos φ	E_a \ E_t	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	4.76197	4.72520	4.61371	4.42312	4.14475	3.76343	3.25263	2.56224	1.58112	-0.03830	-∞
0.82	0.09	5.18049	5.15759	5.05753	4.87652	4.60667	4.23409	3.73473	3.06429	2.12883	0.66238	-3.25866
0.84	0.08	5.66293	5.65496	5.56670	5.39492	5.13243	4.76647	4.27509	3.61867	2.71662	1.35890	-1.51769
0.86	0.07	6.23114	6.23957	6.16366	6.00074	5.74427	5.38244	4.89502	4.24623	3.39584	2.08339	-0.29091
0.88	0.06	6.91914	6.94601	6.88328	6.72881	6.47664	6.11576	5.62734	4.97865	4.10744	2.87186	0.78434
0.90	0.05	7.78344	7.83181	7.78351	7.63695	7.38670	7.02237	6.52622	5.86780	4.99092	3.77442	1.85221
0.92	0.04	8.92706	9.00173	8.96983	8.83037	8.57835	8.20385	7.68980	7.00729	6.10434	4.87436	3.02794
0.94	0.03	10.56435	10.67373	10.66164	10.52775	10.26745	9.87100	9.32148	8.59051	7.62834	6.33742	4.47629
0.96	0.02	13.24844	13.41004	13.42464	13.29275	13.01001	12.56635	11.94408	11.11364	10.02443	8.58132	6.56742
0.98	0.01	19.16871	19.43536	19.49637	19.35371	19.00282	18.43183	17.61944	16.53021	15.10467	13.23536	10.69512
1.00	0.00	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

Table 16. Function X_{13}

	-cos φ	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
-cos φ	E_a \ E_t	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	0.20315	0.23403	0.26833	0.30900	0.35592	0.41225	0.48221	0.57346	0.70205	0.91144	1.49220
0.82	0.09	0.17249	0.19862	0.22812	0.26193	0.30144	0.34875	0.40728	0.48315	0.58891	0.75685	1.13512
0.84	0.08	0.14376	0.16547	0.18994	0.21796	0.25063	0.28965	0.33775	0.39974	0.48532	0.61841	0.89235
0.86	0.07	0.11704	0.13465	0.15448	0.17716	0.20355	0.23500	0.27363	0.32316	0.39094	0.49448	0.69520
0.88	0.06	0.09238	0.10624	0.12183	0.13963	0.16031	0.18489	0.21499	0.25340	0.30555	0.38398	0.52943
0.90	0.05	0.06991	0.08036	0.09211	0.10550	0.12104	0.13947	0.16197	0.19055	0.22907	0.28624	0.38861
0.92	0.04	0.04976	0.05718	0.06551	0.07499	0.08598	0.09898	0.11481	0.13483	0.16164	0.20094	0.26936
0.94	0.03	0.03216	0.03694	0.04230	0.04840	0.05545	0.06378	0.07389	0.08663	0.10358	0.12818	0.16999
0.96	0.02	0.01742	0.02000	0.02289	0.02617	0.02997	0.03444	0.03986	0.04666	0.05566	0.06859	0.09013
0.98	0.01	0.00613	0.00703	0.00805	0.00920	0.01052	0.01208	0.01397	0.01633	0.01943	0.02386	0.03110
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 17. Function χ_{13}

	-cos φ	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
-cos φ	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	5.53595	5.43965	5.30303	5.12427	4.89992	4.62425	4.28812	3.87622	3.36030	2.67608	1.49220
0.82	0.09	5.29349	5.20482	5.07824	4.91218	4.70354	4.44723	4.13512	3.75382	3.27917	2.65904	1.70645
0.84	0.08	5.02968	4.94856	4.83201	4.67867	4.48579	4.24886	3.96069	3.60958	3.17485	2.61382	1.79638
0.86	0.07	4.74095	4.66735	4.56088	4.42038	4.24343	4.02605	3.76192	3.44086	3.04523	2.53987	1.82889
0.88	0.06	4.42245	4.35637	4.26011	4.13267	3.97196	3.77450	3.53478	3.24399	2.88714	2.43531	1.81572
0.90	0.05	4.06718	4.00870	3.92289	3.80891	3.66497	3.48804	3.27341	3.01354	2.69581	2.29652	1.75975
0.92	0.04	3.66448	3.61379	3.53884	3.43895	3.31259	3.15723	2.96886	2.74117	2.46370	2.11726	1.65890
0.94	0.03	3.19646	3.15394	3.09056	3.00579	2.89838	2.76625	2.60611	2.41283	2.17798	1.88643	1.50572
0.96	0.02	2.62848	2.59487	2.54435	2.47651	2.39041	2.28443	2.15604	2.00127	1.81370	1.58202	1.28279
0.98	0.01	1.87166	1.84866	1.81378	1.76677	1.70699	1.63336	1.54418	1.43679	1.30695	1.14729	0.94301
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 18. Function $X_{14} = \chi_{14}$

	-cos φ	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
-cos φ	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	0.44532	0.43011	0.41321	0.39445	0.37356	0.35019	0.32382	0.29366	0.25836	0.21523	0.15633
0.82	0.09	0.38187	0.36894	0.35460	0.33867	0.32095	0.30115	0.27884	0.25337	0.22365	0.18758	0.13962
0.84	0.08	0.32140	0.31062	0.29866	0.28539	0.27064	0.25417	0.23564	0.21451	0.18994	0.16029	0.12156
0.86	0.07	0.26417	0.25540	0.24566	0.23486	0.22287	0.20949	0.19445	0.17733	0.15747	0.13364	0.10294
0.88	0.06	0.21052	0.20359	0.19590	0.18738	0.17793	0.16738	0.15554	0.14210	0.12653	0.10793	0.08425
0.90	0.05	0.16081	0.15557	0.14975	0.14330	0.13616	0.12819	0.11926	0.10913	0.09743	0.08350	0.06594
0.92	0.04	0.11554	0.11181	0.10766	0.10308	0.09800	0.09234	0.08599	0.07881	0.07054	0.06072	0.04845
0.94	0.03	0.07535	0.07294	0.07026	0.06730	0.06402	0.06037	0.05628	0.05166	0.04634	0.04006	0.03225
0.96	0.02	0.04118	0.03987	0.03842	0.03682	0.03505	0.03307	0.03086	0.02837	0.02551	0.02213	0.01797
0.98	0.01	0.01462	0.01416	0.01365	0.01308	0.01246	0.01177	0.01099	0.01012	0.00912	0.00794	0.00649
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 19. Function X_{16}

-cos φ	-cos φ	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	-0.01070	-0.01241	-0.01447	-0.01703	-0.02033	-0.02481	-0.03122	-0.04121	-0.05882	-0.09758	-0.28170
0.82	0.09	-0.00817	-0.00947	-0.01103	-0.01297	-0.01547	-0.01884	-0.02364	-0.03107	-0.04398	-0.07148	-0.16963
0.84	0.08	-0.00605	-0.00701	-0.00816	-0.00958	-0.01142	-0.01387	-0.01737	-0.02272	-0.03192	-0.05101	-0.11151
0.86	0.07	-0.00431	-0.00499	-0.00580	-0.00681	-0.00810	-0.00983	-0.01227	-0.01599	-0.02231	-0.03513	-0.07272
0.88	0.06	-0.00291	-0.00337	-0.00392	-0.00459	-0.00546	-0.00661	-0.00824	-0.01069	-0.01483	-0.02305	-0.04582
0.90	0.05	-0.00184	-0.00212	-0.00247	0.00289	-0.00343	-0.00415	-0.00515	-0.00667	-0.00919	-0.01413	-0.02721
0.92	0.04	-0.00104	-0.00121	-0.00140	-0.00164	-0.00195	-0.00235	-0.00291	-0.00376	-0.00515	-0.00784	-0.01471
0.94	0.03	-0.00051	-0.00058	-0.00068	-0.00079	-0.00094	-0.00113	-0.00140	-0.00180	-0.00246	-0.00371	-0.00681
0.96	0.02	-0.00018	-0.00021	-0.00024	-0.00029	-0.00034	-0.00041	-0.00050	-0.00065	-0.00088	-0.00131	-0.00236
0.98	0.01	-0.00003	-0.00004	-0.00004	-0.00005	-0.00006	-0.00007	-0.00009	-0.00011	-0.00015	-0.00023	-0.00040
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 20. Function X_{16}

-cos φ	-cos φ	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	2.66095	2.18598	1.71840	1.26587	0.83669	0.44034	0.08830	-0.20426	-0.41464	-0.50030	-0.28170
0.82	0.09	3.15794	2.64040	2.12899	1.63159	1.15664	0.71367	0.31403	-0.02762	-0.29014	-0.43640	-0.35822
0.84	0.08	3.72944	3.16288	2.60112	2.05232	1.52515	1.02927	0.57602	0.17971	-0.13964	-0.34891	-0.36733
0.86	0.07	4.39888	3.77464	3.15377	2.54481	1.95674	1.39945	0.88442	0.42579	0.04280	-0.23432	-0.34069
0.88	0.06	5.20244	4.50844	3.81622	3.13484	2.47371	1.84312	1.25480	0.72297	0.26641	-0.08663	-0.28137
0.90	0.05	6.19984	5.41826	4.63669	3.86483	3.11275	2.39129	1.71266	1.09143	0.54632	0.10437	-0.18615
0.92	0.04	7.49897	6.60164	5.70217	4.81125	3.93985	3.09966	2.30372	1.56729	0.90955	0.35717	-0.04551
0.94	0.03	9.32261	8.25968	7.19190	6.13141	5.09056	4.08245	3.12145	2.22418	1.41101	0.70934	0.16201
0.96	0.02	12.24069	10.90653	9.56358	8.22648	6.91105	5.62983	4.40272	3.24793	2.18857	1.25473	0.49093
0.98	0.01	18.48897	16.55703	14.60875	12.66454	10.74494	8.87130	7.06660	5.35661	3.77172	2.35036	1.14665
1.00	0.00	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

Table 21. Function χ_{17}

-cos φ	-cos φ	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	0.01600	0.01841	0.02113	0.02423	0.02785	0.03216	0.03746	0.04428	0.05364	0.06808	0.09810
0.82	0.09	0.01225	0.01409	0.01616	0.01853	0.02128	0.02456	0.02858	0.03372	0.04074	0.05143	0.07223
0.84	0.08	0.00909	0.01045	0.01199	0.01373	0.01577	0.01818	0.02113	0.02490	0.03002	0.03770	0.05206
0.86	0.07	0.00649	0.00746	0.00855	0.00979	0.01123	0.01294	0.01503	0.01769	0.02127	0.02660	0.03624
0.88	0.06	0.00439	0.00505	0.00579	0.00663	0.00760	0.00875	0.01015	0.01193	0.01432	0.01783	0.02404
0.90	0.05	0.00278	0.00319	0.00365	0.00418	0.00479	0.00551	0.00639	0.00750	0.00899	0.01115	0.01490
0.92	0.04	0.00158	0.00182	0.00208	0.00238	0.00273	0.00314	0.00364	0.00426	0.00509	0.00630	0.00835
0.94	0.03	0.00077	0.00088	0.00101	0.00115	0.00132	0.00152	0.00176	0.00206	0.00246	0.00303	0.00399
0.96	0.02	0.00028	0.00032	0.00037	0.00042	0.00048	0.00055	0.00063	0.00074	0.00088	0.00109	0.00142
0.98	0.01	0.00005	0.00006	0.00006	0.00007	0.00008	0.00010	0.00011	0.00013	0.00015	0.00019	0.00025
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 22. Function χ_{17}

-cos φ	-cos φ	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	E_t \ E_a	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10
0.80	0.10	4.42426	3.87515	3.33767	2.81594	2.31433	1.83767	1.39149	0.98255	0.61968	0.31598	0.09810
0.82	0.09	4.28874	3.76352	3.24904	2.74918	2.26802	1.81006	1.38043	0.98533	0.63275	0.33426	0.11114
0.84	0.08	4.13042	3.63122	3.14189	2.66604	2.20747	1.77035	1.35940	0.98030	0.64025	0.34947	0.12559
0.86	0.07	3.94562	3.47943	3.01324	2.56389	2.13038	1.71655	1.32672	0.96605	0.64102	0.36062	0.13963
0.88	0.06	3.72938	3.29019	2.85911	2.43921	2.03369	1.64604	1.28019	0.94078	0.63358	0.36648	0.15194
0.90	0.05	3.47475	3.07073	2.67392	2.28709	1.91313	1.55519	1.21678	0.90201	0.61599	0.36554	0.16115
0.92	0.04	3.17125	2.80714	2.44930	2.10020	1.76239	1.43864	1.13204	0.84618	0.58546	0.35572	0.16566
0.94	0.03	2.80163	2.48394	2.17154	1.86654	1.57113	1.28768	1.01882	0.76756	0.53763	0.33384	0.16326
0.96	0.02	2.33295	2.07164	1.81454	1.56335	1.31985	1.08593	0.86370	0.65560	0.46454	0.29432	0.15035
0.98	0.01	1.68199	1.49588	1.31265	1.13352	0.95972	0.79259	0.63357	0.48436	0.34696	0.22395	0.11897
1.00	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

The remaining functions are given by

$$X_1 = \Phi_1 \psi_{31}$$

$$X_2 = \tfrac{1}{2} \Phi_2 \psi_{31}$$

$$X_6 = \tfrac{1}{2} \Phi_1 \psi_8$$

$$X_7 = \tfrac{1}{4} \Phi_2 \psi_8$$

$$X_{11} = \Phi_{13} \psi_{31}$$

$$X_{15} = \tfrac{1}{2} \Phi_{13} \psi_8$$

$$X_{18} = \chi_5$$

$$\chi_1 = \psi_1 \Phi_{31}$$

$$\chi_2 = \tfrac{1}{2} \psi_2 \Phi_{31}$$

$$\chi_6 = \tfrac{1}{2} \psi_1 \Phi_8$$

$$\chi_7 = \tfrac{1}{4} \psi_2 \Phi_8$$

$$\chi_{11} = \psi_{13} \Phi_{31}$$

$$\chi_{15} = \tfrac{1}{2} \psi_{13} \Phi_8$$

$$\chi_{18} = X_5$$

APPENDIX II

Binary and Ternary Solutions and Appropriate Stability Tests

General forms of solution for binary and ternary calculations are described in Sections 3 and 4 of this report. These are amplified here into a detailed form which will enable a computor to obtain from the equations of motion solutions for critical speed and frequency. The process is taken from the stage where values of the basic a, γ, b, and c coefficients have been obtained.

In addition to the determination of critical speed it is sometimes necessary to decide upon which side of a critical boundary the stable and unstable regions lie. More explicitly, if the critical speed has been determined for a range of values of some variable parameter, say a structural stiffness, then the curve obtained by plotting critical speed against the parameter is the critical boundary, representing steady sinusoidal oscillation with constant amplitude. Points lying off this boundary represent either stable or unstable conditions with decreasing or increasing amplitude respectively, and it may not always be obvious which side of the boundary represents the stable and which the unstable condition. In such cases stability tests are available to define these regions. Each of the solutions given below is accompanied by an appropriate stability test.

1. Direct Iterative Solution for Binary

$$\delta_{rs} = a'_{rs}\lambda^2 + b_{rs}\lambda + c_{rs}$$

where

$$a'_{rs} = a_{rs} + \gamma_{rs}, \quad \text{and} \quad \lambda = i\,\omega_m .$$

Coefficients b and c are calculated for an assumed value of ω_m.

The determinantal equation obtained directly from the equations of motion is

$$\begin{vmatrix} \delta_{11} + e_{11}, & \delta_{12} \\ \delta_{21}, & \delta_{22} + e_{22} \end{vmatrix} = 0 \tag{1}$$

and is expanded in the form

$$p_0\lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4 = 0 \tag{2}$$

The notation (x, y) is adopted to represent the sum of the distinct determinants of type (1), which can be made with all possible permutations of x and y taken together, each being associated with a row of the determinant. In the general case $x \neq y$ there are two permutations, xy and yx, and therefore

$$(x, y) = \begin{vmatrix} x_{11} & x_{12} \\ y_{21} & y_{22} \end{vmatrix} + \begin{vmatrix} y_{11} & y_{12} \\ x_{21} & x_{22} \end{vmatrix}$$

In the specific case $x = y = z$ there is only one permutation, zz, and therefore

$$(z, z) = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}$$

Using this notation, the values of the p coefficients are

$$p_o = (a', a')$$

$$p_1 = (a', b)$$

$$p_2 = A + Be_{22}$$

$$p_3 = C + De_{22}$$

$$p_4 = E + Fe_{22} + Re_{22}^2$$

where

$$A = (a', c) + (b, b)$$

$$B = a'_{11} + Ra'_{22}$$

$$C = (b, c)$$

$$D = b_{11} + Rb_{22}$$

$$E = (c, c)$$

$$F = c_{11} + Rc_{22}$$

and $$R = \text{stiffness ratio} = \frac{\ell_\phi}{m_\theta} \cdot \frac{c_m^2}{\ell^2}$$

The test function, obtained by equating the real and imaginary parts of (2) to zero and eliminating ω_m, when expanded gives the following quadratic in e_{22}.

$$(p_1 BD - p_o D^2 - p_1{}^2 R)\, e_{22}^2 + (p_1 \overline{AD + BC} - 2p_o CD - p_1{}^2 F)\, e_{22} + (p_1 AC - p_o C^2 - p_1{}^2 E) = 0 \qquad (3)$$

Equation (3) is solved for e_{22} and the frequency obtained from $\omega_m^2 = p_3/p_1$. If ω_m agrees reasonably well with the assumed value, the critical speed is then obtained directly from

$$V = \frac{1}{c_m} \sqrt{\frac{m_\theta}{\rho \ell e_{22}}}$$

Stability Test

The critical condition is pre-supposed in the above solution by taking $\lambda = i\omega_m$ in equation (2). The motion is then proportional to e^{ipt}, that is sinusoidal with time. Equation (2) can however equally well represent the general condition in which any λ root has the form $\lambda = u + i\omega_m$. In the critical condition the speed and the value of the variable parameter considered make the p coefficients such that a solution for λ is obtained with $u = 0$. With slightly different values of either speed or parameter a solution would be obtained with $u \neq 0$, the resulting oscillation being stable or unstable according to whether u is negative or positive respectively.

A second solution could therefore be performed using a slightly different value of speed or parameter. The stability would be indicated by the sign of the resulting value of u, and the region labelled accordingly.

Standard stability tests have however been devised which avoid the necessity for a complete solution.

The full set of conditions for stability in this case are

(a) all coefficients p must be positive

(b) the test determinant T_3 must be positive.

$$T_3 = \begin{vmatrix} p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 \\ 0 & p_4 & p_3 \end{vmatrix}$$

The procedure is therefore to examine these conditions for a slightly different value of speed or parameter. For moderate departures from the critical condition it will generally be found that condition (a) is still satisfied and the definition of stability therefore rests upon the sign of T_3, which is zero in the critical condition.

2. Indirect Non-Iterative Solution for Binary

In this case the form

$$\delta_{rs} = \alpha_{rs} + i\,\beta_{rs}$$

is used, where

$$\alpha_{rs} = -(a_{rs} + \gamma_{rs})\omega_m^2 + c_{rs} \quad \text{and} \quad \beta_{rs} = \omega_m b_{rs}$$

Coefficients α and β are calculated for a given value of ω_m.

The determinantal equation (1) is expanded in the form

$$|\delta| + \delta_{22}\, e_{11} + \delta_{11}\, e_{22} + e_{11}\, e_{22} = 0 \tag{4}$$

which, when real and imaginary parts are equated to zero, gives the two equations,

$$\left.\begin{aligned} R_o + R_1 e_{11} + R_2 e_{22} + R_3 e_{11} e_{22} &= 0 \\ S_o + S_1 e_{11} + S_2 e_{22} + S_3 e_{11} e_{22} &= 0 \end{aligned}\right\} \tag{5}$$

where

$$R_o = (\alpha,\alpha) - (\beta,\beta)$$

$$R_1 = \alpha_{22}, \quad R_2 = \alpha_{11}, \quad R_3 = 1$$

$$S_o = (\alpha,\beta)$$

$$S_1 = \beta_{22}, \quad S_2 = \beta_{11}, \quad S_3 = 0$$

Eliminating

$$- e_{11} = \frac{R_o + R_2 e_{22}}{R_1 + R_3 e_{22}} = \frac{S_o + S_2 e_{22}}{S_1 + S_3 e_{22}} \tag{6}$$

then gives the following quadratic in e_{22}:

$$(R_2 S_3 - R_3 S_2) e_{22}^2 + (R_o S_3 - R_3 S_o + R_2 S_1 - R_1 S_2) e_{22} + (R_o S_1 - R_1 S_o) = 0 \tag{7}$$

For the given value of ω_m, equation (7) is solved for e_{22}, e_{11} is obtained from equation (6), and hence the stiffness ratio $R = \frac{e_{11}}{e_{22}}$. The whole process is then repeated for several values of ω_m and finally R is plotted against say e_{22}. From the curve the value of e_{22} corresponding to the actual value of R is obtained, and hence the critical speed from

$$V = \frac{1}{c_m} \sqrt{\frac{m_\theta}{\rho\, \ell\, e_{22}}}$$

Stability Test

The standard stability test given for the direct iterative solution could be applied, but this would involve a separate determination of the p coefficients. It is more convenient to use a test which is consistent with the type of solution adopted, and for the indirect non-iterative solution the following test has been suggested.

The principle of the test is to repeat the solution for a given value of ω_m but including an arbitrary small amount of structural damping. Values of e_{22} and R obtained from the original solution will be represented by some point on the curve of e_{22} plotted against R (the critical boundary). From the repeat solution with structural damping slightly different values of e_{22} and R will be obtained, giving a point close to but lying off the critical boundary. This new point represents the critical condition with structural damping present, and intuitively it follows that the side of the boundary on which the new point lies must be the unstable region for the original condition without structural damping.

Force due to structural stiffness is proportional to displacement and force due to structural damping is proportional to velocity. For the co-ordinate q_1, for instance, the stiffness force is proportional to $e_{11}q_1$, and the damping force proportional to $\dot{q}_1$, or to $i\omega_m q_1$.

The net force due to stiffness and damping is therefore proportional to $(e_{11} + i\omega_m k)q_1$, k being an appropriate constant. For an arbitrary amount of structural damping this may be written as $e_{11}(1 + i\mu)q_1$, μ being an arbitrary quantity representing the damping. Changing from the undamped to an arbitrarily damped condition can therefore be represented by multiplying each stiffness coefficient by $(1 + i\mu)$.

With structural damping equation (4) then becomes

$$|\delta| + \delta_{22}e_{11}(1 + i\mu) + \delta_{11}e_{22}(1 + i\mu) + e_{11}e_{22}(1 + i\mu)^2 = 0 \qquad (8)$$

and the coefficients in equations (5) are modified as follows:-

R_o and S_o are unaltered

R_1 becomes $R_1 - \mu S_1$

R_2 becomes $R_2 - \mu S_2$

R_3 becomes $R_3(1 - \mu^2) - 2\mu S_3$

S_1 becomes $S_1 + \mu R_1$

S_2 becomes $S_2 + \mu R_2$

S_3 becomes $S_3(1 - \mu^2) + 2\mu R_3$

For a given ω_m and a small arbitrary value of μ equations (6) and (7) are re-solved for e_{11}, e_{22} and R using the modified coefficients above.

The location of the resulting point (e_{22},R) relative to the original critical boundary then determines the unstable region for the condition without structural damping.

3. Direct Iterative Solution for Ternary ($e_{33} = 0$)

The determinantal equation obtained directly from the equations of motion is

$$\begin{vmatrix} \delta_{11}+e_{11}, & \delta_{12}, & \delta_{13} \\ \delta_{21}, & \delta_{22}+e_{22}, & \delta_{23} \\ \delta_{31}, & \delta_{32}, & \delta_{33} \end{vmatrix} = 0 \qquad (9)$$

and is expanded in the form

$$p_o\lambda^6 + p_1\lambda^5 + p_2\lambda^4 + p_3\lambda^3 + p_4\lambda^2 + p_5\lambda + p_6 = 0 \qquad (10)$$

Coefficients b and c are calculated for an assumed value of ω_m.

The notation adopted for the binary is extended, (x, y, z) representing the sum of the distinct determinants of type (9) which can be made with all possible permutations of x, y and z taken together, each being associated with a row of the determinant. In the general case $x \neq y \neq z$ there are six permutations, xyz xzy yzx yxz zxy and zyx, so that

$$(x, y, z) = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ y_{21} & y_{22} & y_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ z_{21} & z_{22} & z_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix} + \text{etc.}$$

When two of the three elements are equal, as in (x, x, y), there are only three permutations, xxy xyx and yxx, so that

$$(x, x, y) = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ y_{21} & y_{22} & y_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} + \begin{vmatrix} y_{11} & y_{12} & y_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}$$

When all three elements are equal, as in (x, x, x), there is only one permutation xxx, and therefore

$$(x, x, x) = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}$$

In addition, $(x, y)_{11}$ and $(x, y)_{22}$ are used to represent similar permutations of x and y with respect to the minors of δ_{11} and δ_{22} respectively in $|\delta|$. For instance

$$(x, y)_{11} = \begin{vmatrix} x_{22} & x_{23} \\ y_{32} & y_{33} \end{vmatrix} + \begin{vmatrix} y_{22} & y_{23} \\ x_{32} & x_{33} \end{vmatrix}$$

and

$$(x, x)_{22} = \begin{vmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{vmatrix}$$

Using this notation, the values of the p coefficients are

$$p_o = (a', a', a')$$

$$p_1 = (a', a', b)$$

$$p_2 = (a', a', c) + (a', b, b) + e_{11}(a', a')_{11} + e_{22}(a', a')_{22}$$

$$p_3 = (a', b, c) + (b, b, b) + e_{11}(a', b)_{11} + e_{22}(a', b)_{22}$$

$$p_4 = (a', c, c) + (b, b, c) + e_{11}\{(a', c)_{11} + (b, b)_{11}\}$$
$$+ e_{22}\{(a', c)_{22} + (b, b)_{22}\} + a'_{33}\, e_{11}\, e_{22}$$

$$p_5 = (b, c, c) + e_{11}(b, c)_{11} + e_{22}(b, c)_{22} + b_{33}\, e_{11}\, e_{22}$$

$$p_6 = (c, c, c) + e_{11}(c, c)_{11} + e_{22}(c, c)_{22} + c_{33}\, e_{11}\, e_{22}$$

Equating real and imaginary parts of (10) to zero gives the two equations

$$- p_o\omega_m^6 + p_2\omega_m^4 - p_4\omega_m^2 + p_6 = 0 \qquad (11)$$

$$p_1\omega_m^4 - p_3\omega_m^2 + p_5 = 0 \qquad (12)$$

By eliminating ω_m from these equations and substituting $e_{11} = Re_{22}$ a sextic in e_{22} can be formed. Direct solution of this is laborious and therefore rarely used. Instead, equations (11) and (12) can be solved indirectly. For a given value of e_{22}, and hence of e_{11}, the p coefficients can be calculated and equation (12) solved as a quadratic in ω_m^2, whose value is then substituted in equation (11). Repeating the process over a range of values of e_{22}, the value for which the left-hand side of equation (11) is zero can be found by interpolation. If the associated value of ω_m agrees reasonably well with the value originally assumed for the calculation of the b and c coefficients, then the critical speed is given directly by

$$V = \frac{1}{c_m}\sqrt{\frac{m_\theta}{\rho\, \ell\, e_{22}}}$$

Stability Test

For the standard test the full set of conditions for stability of the sextic (10) are

(a) coefficients p_o, p_1 and p_6 must be positive

(b) the test determinants T_2, T_3, T_4 and T_5 must be positive.

$$T_2 = \begin{vmatrix} p_1 & p_o \\ p_3 & p_2 \end{vmatrix} \qquad T_3 = \begin{vmatrix} p_1 & p_o & 0 \\ p_3 & p_2 & p_1 \\ p_5 & p_4 & p_3 \end{vmatrix}$$

$$T_4 = \begin{vmatrix} p_1 & p_o & 0 & 0 \\ p_3 & p_2 & p_1 & p_o \\ p_5 & p_4 & p_3 & p_2 \\ 0 & p_6 & p_5 & p_4 \end{vmatrix} \qquad T_5 = \begin{vmatrix} p_1 & p_o & 0 & 0 & 0 \\ p_3 & p_2 & p_1 & p_o & 0 \\ p_5 & p_4 & p_3 & p_2 & p_1 \\ 0 & p_6 & p_5 & p_4 & p_3 \\ 0 & 0 & 0 & p_6 & p_5 \end{vmatrix}$$

The procedure, as for the binary, is therefore to examine these conditions for a value of speed or parameter slightly different from the critical. The stability will generally be determined by the sign of T_5, which is zero in the critical condition.

4. <u>Indirect Non-Iterative Solution for Ternary</u> ($e_{33} = 0$)

As for the binary, coefficients α and β are calculated for a given value of ω_m.

The determinantal equation (9) is expanded in the form

$$|\delta| + \Delta_{11} e_{11} + \Delta_{22} e_{22} + \delta_{33} e_{11} e_{22} = 0 \qquad (13)$$

which, when real and imaginary parts are equated to zero, gives the two equations (5) but in this case with

$$R_o = (\alpha, \alpha, \alpha) - (\alpha, \beta, \beta)$$

$$R_1 = (\alpha, \alpha)_{11} - (\beta, \beta)_{11}$$

$$R_2 = (\alpha, \alpha)_{22} - (\beta, \beta)_{22}$$

$$R_3 = \alpha_{33}$$

$$S_o = (\alpha, \alpha, \beta) - (\beta, \beta, \beta)$$

$$S_1 = (\alpha, \beta)_{11}$$

$$S_2 = (\alpha, \beta)_{22}$$

$$S_3 = \beta_{33}$$

Using equations (6) and (7), the solution then proceeds exactly as for the binary.

Stability Test

Applying the structural damping test, the solution is repeated with e_{11} and e_{22} in equation (13) each multiplied by $(1 + i\mu)$. The same modifications are made to the coefficients of equation (5) as in the binary case, but using of course the original values appropriate to the ternary as given above.

For a given ω_m and a small arbitrary value of μ equations (6) and (7) are re-solved using the modified coefficients. The location of the resulting point (e_{22}, R) relative to the original critical boundary then determines the unstable region for the condition without structural damping.

APPENDIX III

Interpretation and Use of Resonance Test Results

The fact that a relationship frequently exists between the still air modes (i.e. normal modes) of vibration of an aircraft and its flutter characteristics has been appreciated for some time. In recent years this appreciation has been acknowledged by the requirement for resonance tests to be made before flight on each new prototype aircraft, as a safety precaution against flutter.

The technique of the tests, as described in R & M 2155[1], is now fairly generally understood but there are still many widespread misconceptions as to the practical uses of the results. The resonance test results cannot, at the present stage, be interpreted so as to supply a complete picture of the flutter characteristics of an aircraft, nor does the fact that the interpreter obtains a negative result from the analysis necessarily imply that the aircraft will be free from flutter. In the light of past experience, from a careful consideration of the results it is often possible to assess the likelihood of the aircraft avoiding flutter trouble, and if a flutter incident or accident does occur the results may provide an immediate indication as to the best cure.

In what follows the salient points of the resonance test analysis and the application of the results are discussed; and, in particular, the application to theoretical investigations is described and exemplified by a sample normal mode calculation on a hypothetical aircraft.

Analysis of Resonance Test Results

In recent years experience has been to the effect that main and auxiliary control surfaces almost invariably play the predominant part in flutter troubles that occur in practice and as a result the usual practice in the analysis is to concentrate on phenomena which are known to be relevant to the flutter of these items. However it is quite conceivable that with the radical changes of design now taking place the emphasis in the future may be on the flutter of the main structure, and therefore for any particular analysis all aspects must be kept in mind.

The two major features indicative of possible control surface flutter that are looked for in resonance tests may be classed broadly as

(a) ineffective mass balance, and

(b) a proximity of any two of the natural frequencies of the main and auxiliary controls and the aircraft structure.

Since the purpose of mass balancing is to eliminate inertia couplings between the control surface and main surface motions it should be strictly related to the actual modes experienced in flight when a vibration occurs. If the mass balancing is effective the vibration is damped and flutter is avoided. Mass balancing criteria given in A.P.970 are related to assumed modes of a simple type and are to be regarded as first approximations only. Normal modes as obtained from resonance tests represent on the whole a much closer approximation and provide a useful check on the mass balancing system adopted. For aircraft in which concentrated masses are used for mass balance the resonance test results are analysed for modes in which a

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[1] W.G. Molyneux and E.G. Broadbent. "Ground Resonance Testing of Aircraft".

balance weight is in close proximity to a nodal line. A balance weight on a nodal line serves no useful purpose in that particular mode and accordingly the greater the number of balance weights the less is the likelihood of trouble from this cause, for in any particular mode in which there is a loss of the effectiveness of one weight there might quite possibly be an increase in the effectiveness of the others. The single mass is that most likely to give trouble, and the likelihood of trouble is enhanced when the balance weight is remote from the surface in such cases for it is possible for the weight to act in an anti-balance sense by virtue of a nodal line existing between the weight and the surface.

Certain of the phenomena leading to tab flutter may also be classified under (a). Geared and trimmer tabs frequently carry no mass balance on the assumption that no degree of freedom separate from that of the main control is possible and on such a system any resonance mode in which there is excessive rotation of the tab relative to the main control is at once suspected. Such rotation may be due to backlash or undue flexibility in the tab circuit.

Modes under case (b) above have been definitely identified in a number of cases as being a contributory cause of flutter trouble and it appears that frequency proximity may lead to flutter even on a fully mass balanced system. Phenomena of this type are apparent from the resonance test results for it is general practice to obtain "amplitude-frequency" curves for the control surfaces in addition to those of the main structure, and from these curves an estimate of the proximity of the relevant frequencies may be obtained.

Spring tabs are in a special category since, because of their intrinsic freedom relative to the main control, a degree of mass balance of the tab is normally required (spring tabs in fact need special treatment in this respect and the optimum weight of mass balance may well be zero in certain cases). Troubles associated with spring tabs may therefore occur under (a) or (b). The same is of course true of the main control when the stiffness of the control circuit is considered. In the case of the main control, measurements on the control column will distinguish a resonance of the control circuit from bodily movement but it is not so easy to distinguish between the two for a tab. In any case coupled rotation of any kind is suspected since whatever the cause the rotation is likely to influence the flutter characteristics.

<u>Action Following Analysis</u>

The mere fact that the resonance test analysis indicates a susceptibility of the aircraft to some particular type of flutter is not necessarily conclusive. It may be that flutter, if it occurs at all, is at a speed beyond the range of the aircraft, or the mass balance may still be sufficient to render the system immune from flutter despite some loss in effectiveness; or whatever has been suspect may prove after all to be adequate. A possible approach to the problem would be immediately to modify the aircraft so as to remove the adverse resonance characteristics, but this would certainly lead to many unnecessary modifications if applied universally. However, this approach has its applications in cases where flying is required urgently and the risk of flutter cannot be tolerated, and in particular for cases of flutter that have occurred in which the general form of the flutter is known. For the general case the most satisfactory procedure is to examine particular suspected cases on a theoretical basis, as a result of which suitable modifications may if necessary be made.

As mentioned earlier it is quite possible that, despite all the precautions taken prior to flight, flutter may still occur on the aircraft. The failure of resonance test results to forecast failure in such cases is a measure of the present undeveloped state of the analysis, but each case of flutter that occurs adds to the fund of knowledge and extends the range of the analysis. Developments in analysis result for instance when flutter occurs in which the modes involved may be of a type for which no previous experience exists to demonstrate susceptibility to flutter. Such modes would not in the first instance appear significant. Proximity of the resonance frequencies of components is another feature about which there is much to be learned, for it is difficult at the moment to know what degree of proximity is to be regarded as serious. However, when flutter troubles occur, the resonance test results will, in many cases, give an indication of the source of the trouble and will indicate the best line of attack for effecting a cure. When the flutter is of a form too complicated for the test results to give any direct indication of the best line of attack the normal modes are nevertheless of considerable value in any theoretical investigations that are made.

Application of the Results to Theoretical Investigations

When theoretical investigations are undertaken, either prior to flight as a result of resonance test indications of flutter susceptibility or after an incident has occurred in flight, the normal modes obtained from the resonance tests are generally used for the calculations.

As explained in Section 2 of this report, flutter investigations are normally made by restricting the calculation to a specified number of degrees of freedom of the aircraft, and to obtain reliable results these must be chosen such that when coupled together with the appropriate amplitude and phase relationships (to be determined implicitly in the calculation) the final motion agrees closely with the true physical motion under flutter conditions. If the modes are well chosen a good answer will be obtained in quite a small number of degrees of freedom, but if the modes are ill chosen that number may be greatly increased, and when it is realised that the computational labour increases roughly as the factorial of the number of degrees of freedom chosen it will be appreciated that a good choice of modes becomes a matter of prime importance.

It is still very much undecided as to whether normal modes will in general permit greater accuracy than the equivalent approach using "arbitrary" modes, but for certain specified cases the resonance modes are a virtual necessity. These occur for instance when resonance tests give a mode in which the nodal line is suspiciously close to a mass balance weight; for then the obvious flutter condition to investigate is one having a mode similar to the resonance mode, which is therefore taken as one of the degrees of freedom. In cases of this kind the flutter frequency is often in close agreement with the frequency of the normal mode. If an arbitrary mode is chosen in such an instance there is a greater likelihood of a large error in nodal shape, and the associated stiffness is particularly unreliable as it depends on the second differential of the mode. When simple arbitrary modes of the fundamental type are used the associated stiffnesses are usually not even related to the mode itself but are represented by static stiffnesses appropriate to the application of a concentrated load at some "reference" station. With a normal mode the stiffness is given simply and accurately by the measured frequency and the inertia characteristics.

Other respective advantages of the two methods are of small importance. On the one hand the normal mode approach eliminates the cross-inertias and cross-stiffnesses (except, of course, for the control surface degree of freedom) whereas the simple modes render the aerodynamic treatment somewhat easier.

As an illustration of the type of investigation carried out with normal modes a sample calculation is given at the end of this Appendix. The investigation is applied to a hypothetical aircraft on the presumption of a suspected inefficiency of the elevator geared mass balance weight (which from Fig.1 is seen to be close to a node in the fuselage), and the calculation is based on only two degrees of freedom, namely, the particular normal mode and elevator rotation. But although the treatment of the normal mode is typical of current practice it must not be thought that the example is typical of a flutter calculation as a whole. The scope of the calculation (for simplicity) has been restricted far too much to be used for direct application, and in practice at least three degrees of freedom would have to be used for a calculation of this sort. The degrees of freedom normally considered for symmetric elevator flutter are:-

(1) First normal mode involving fuselage bending

(2) Second " " " " "

(3) Elevator rotation

(4) Pitch of the whole aircraft

(5) Vertical translation of the whole aircraft.

Of these five the last can usually be neglected as its effect upon the flutter speed will usually be small. In some cases a further simplification may be effected by making use of the fact that for a conventional aircraft the wing motion associated with (1) and (2) will be almost pure flexure which will be heavily damped in flight. The flutter condition will therefore be that in which this damping is a minimum, i.e. modes (1) and (2) will combine to give as little net wing motion as possible. In the calculation below the full wing motion is assumed and the fact that the system still possesses a fairly low flutter speed may be explained by the fact that a very bad case has been chosen, with a heavy elevator and almost zero effectiveness from the mass balance weight.

<u>Sample Normal Mode Calculation</u>

The ensuing worked example has been carried out on a hypothetical aircraft for which certain assumptions have been made to simplify the arithmetic. The wing and tailplane are both assumed rectangular and in general the modes are supposed to be expressible as simple algebraic functions. This will in fact be very nearly true for fundamental modes of vibration even in practice though the inertia data will often be available in such form as to make analytical integrations for the inertia coefficients not very easy. Diagrams of the assumed (normal) modes of vibration are given in Fig.1.

The complete normal mode of the aircraft may be expressed as

$$z = \ell f_j(\eta)\, q_1 \qquad j = 1,2,3$$

$$\alpha = F_j(\eta)\, q_1 \qquad j = 1,2$$

q_1 is the generalised co-ordinate of the degree of freedom corresponding to the normal mode, so that ℓq_1 is the amplitude at the reference section where $f(\eta)$ is unity. For convenience the wing tip is chosen as the reference section and ℓ is put equal to one foot. $f_1(\eta)$, $f_2(\eta)$ and $f_3(\eta)$ represent the flexural modes of the wing, tailplane and fuselage respectively (all corresponding to unity at the wing tip). Similarly $F_1(\eta)$, $F_2(\eta)$ represent the torsional modes of the wing and tailplane respectively, corresponding to a unit value of $f(\eta)$ at the wing tip.

For a torsionally rigid elevator the local elevator angle is given by

$$\xi = \xi_o + \alpha_o - \alpha$$

where ξ_0 and α_0 are the angles of the elevator and tailplane respectively as measured at the elevator lever section.

Hence

$$\xi = q_2 + (F_2' - F_2)\, q_1$$

where $\alpha_0 = F_2' q_1$, $\xi_0 = q_2$.

The vertical displacement of the mass balance is

$$z - r\beta = \ell f_3' q_1 - r(q_2 + F_2' q_1)$$

where z is the displacement of the mass balance hinge, β is the rotation of the mass balance arm relative to space, and r is the effective mass balance arm. The value for β depends on the gear ratio between the elevator and the mass balance, which has in this case been taken as unity.

As in equation (1.7) of Section 3, if $p/2\pi$ is the flutter frequency then

$$p^2 = \omega_m^{\ 2} \frac{V^2}{c_m^{\ 2}}$$

where ω_m is the mean frequency parameter corresponding to the wing mean chord c_m. For the wing the local frequency parameter $\omega_w = p\,\frac{c_w}{V}$. For the tailplane the local frequency parameter $\omega_t = p\,\frac{c_t}{V}$.

If $$\lambda = i\omega_m = i\,p\,\frac{c_m}{V}$$

then $$i\omega_w = \lambda\,\frac{c_w}{c_m}$$

and $$i\omega_t = \lambda\,\frac{c_t}{c_m}$$

Inertia Coefficients

Using the same notation as in Section 3 the equation for the total kinetic energy may be constructed as

$$L_w = \rho c_w V^2 (-\omega_w^2 \ell_{\ddot{z}} + i\omega_w \ell_{\dot{z}} + \ell_z) \frac{z_w}{c_w} \quad **$$

$$+ \rho c_w V^2 (-\omega_w \ell_{\ddot{\alpha}} + i\omega_w \ell_{\dot{\alpha}} + \ell_\alpha) \alpha_w$$

and on the tailplane by

$$L_t = \rho c_t V^2 (-\omega_t^2 \ell_{\ddot{z}}' + i\omega_t \ell_{\dot{z}}' + \ell_z') \frac{z_t}{c_t} \quad **$$

$$+ \rho c_t V^2 (-\omega_t^2 \ell_{\ddot{\alpha}}' + i\omega_t \ell_{\dot{\alpha}}' + \ell_\alpha') \alpha_t$$

$$+ \rho c_t V^2 (-\omega_t^2 \ell_{\ddot{\xi}}' + i\omega_t \ell_{\dot{\xi}}' + \ell_\xi') \xi$$

Re-writing in terms of the mean frequency parameter $\lambda = i\omega_m$

$$L_w = \rho c_w V^2 \left\{\lambda^2 \left(\frac{c_w}{c_m}\right)^2 \ell_{\ddot{z}} + \lambda \left(\frac{c_w}{c_m}\right) \ell_{\dot{z}} + \ell_z\right\} \frac{z_w}{c_w}$$

$$+ \text{etc.}$$

$$L_t = \rho c_t V^2 \left\{\lambda^2 \left(\frac{c_t}{c_m}\right)^2 \ell_{\ddot{z}}' + \lambda \left(\frac{c_t}{c_m}\right) \ell_{\dot{z}}' + \ell_z\right\} \frac{z_t}{c_t}$$

$$+ \text{etc.}$$

where the dashed derivatives refer to the tailplane and the undashed to the wing.

The moment about the leading edge M and the elevator hinge moment H, may be similarly expressed.

Proceeding as in Section 3 the aerodynamic coefficients may be obtained. The aerodynamic stiffness coefficients are as follows:

** z_w and z_t are leading edge displacements of wing and tailplane.

$$c_{11} = \int_{wing} \frac{\ell_z}{c_m^2} (\ell f_1 - h_w c_w F_1)^2 \, d\eta$$

$$+ \int_{wing} \left(\frac{c_w}{c_m}\right) \left(\frac{\ell_\alpha - m_z}{c_m}\right) (\ell f_1 - h_w c_w F_1) F_1 \, d\eta$$

$$+ \int_{wing} - m_\alpha F_1^2 \left(\frac{c_w}{c_m}\right)^2 d\eta$$

$$+ \frac{s_t}{s_w} \int_{tailplane} \frac{\ell_z'}{c_m^2} (\ell f_2 - h_t c_t F_2)^2 \, d\eta$$

$$+ \frac{s_t}{s_w} \int_{tailplane} \left(\frac{c_t}{c_m}\right) \left(\frac{\ell_\alpha' - m_z'}{c_m}\right) (\ell f_2 - h_t c_t F_2) F_2 \, d\eta$$

$$+ \frac{s_t}{s_w} \int_{tailplane} m_\alpha' F_2^2 \left(\frac{c_t}{c_m}\right)^2 d\eta$$

$$c_{12} = \frac{s_t}{s_w} \int_{tailplane} \left(\frac{c_t}{c_m}\right) \frac{\ell_\xi'}{c_m} (\ell f_2 - h_t c_t F_2) \, d\eta$$

$$- \frac{s_t}{s_w} \int_{tailplane} m_\xi' F_2 \left(\frac{c_t}{c_m}\right)^2 d\eta$$

$$c_{21} = - \frac{s_t}{s_w} \int_{tailplane} \left(\frac{c_t}{c_m}\right) \frac{h_z'}{c_m} (\ell f_2 - h_t c_t F_2) \, d\eta$$

$$- \frac{s_t}{s_w} \int_{tailplane} h_\alpha' F_2 \left(\frac{c_t}{c_m}\right)^2 d\eta$$

$$c_{22} = - \frac{s_t}{s_w} \int_{tailplane} h_\xi' \left(\frac{c_t}{c_m}\right)^2 d\eta$$

$h_w c_w$, $h_t c_t$ are the distances from the reference axis to the leading edge for the wing and tailplane respectively.

As in Section 3, the b and γ coefficients for any given order are obtained from the c coefficients of the same order by including the appropriate factors $\frac{c}{c_m}$, $\left(\frac{c}{c_m}\right)^2$ within the integrals and using appropriate damping and virtual inertia derivatives.

For the hypothetical mode of Fig. 1 the main structural distortions are expressed as mathematical functions, and the integrals may be determined exactly. In practice the integrals would be determined by some approximate method, and usually by a summation on Simpson's rule.

The values of the various constants are as follows:-

s_w = 20 ft

s_t = 7.5 ft

s_f = 20 ft

$c_m = c_w$ = 8 ft

c_t = 5 ft

$\bar{x}_2$ = 1.05 ft

K_2 = 1.25 ft

$\bar{x}_3$ = 0.5 ft

K_3 = 1.0 ft

x_h = 1.75 ft

$h_w c_w$ = 2 ft

$h_t c_t$ = 1.25 ft

r = 2 ft

The mass distributions m_1, m_2, m_3, m_4 are as shown in Fig. 2.

$f_1(\eta) = 1.5\,\eta^2 - 0.5$ $\qquad F_1(\eta) = 0$) Fuselage slope

$f_2(\eta) = 0.72\,\eta^2 + 0.78$ $\qquad F_2(\eta) = -0.128$) at $\frac{1}{4}$ chord positions

$$f_3(\eta) = \begin{cases} 1.28\,\eta^2 - 0.5\,, & \eta = 1.0 \text{ to } \eta = 0 \\ 2.0\,\eta^2 - 0.5\,, & \eta = 0 \text{ to } \eta = -0.625 \end{cases}$$

In the determination of the derivatives the elevator chord aft of the hinge is 2.0 ft and the elevator chord forward of the hinge is 0.5 ft. A value for the frequency parameter of 0.5 has been assumed.

The absolute (theoretical) values of the derivatives have been factored as follows:-

Absolute value of $h_{\dot{\xi}}$ factored by 0.65

" " " $\ell_{\dot{z}}$ factored by 0.75

" " " all stiffness derivatives factored by 0.5.

These factors should not be regarded too seriously as they are based on a single comparison[1] made with experimental values obtained by Frazer and Duncan many years ago. As mentioned in Section 2, further work is required in this direction.

The derivatives used in the investigation can now be obtained as:

Wing

$\ell_{\ddot{z}} = 0.7854$;	$\ell_{\ddot{\alpha}} = 0.3927$;	$-m_{\ddot{z}} = 0.3927$;	$-m_{\ddot{\alpha}} = 0.2209$
$\ell_{\dot{z}} = 1.6321$;	$\ell_{\dot{\alpha}} = 2.1266$;	$-m_{\dot{z}} = 0.5440$;	$-m_{\dot{\alpha}} = 0.7062$
$\ell_{z} = 0.1455$;	$\ell_{\alpha} = 1.1972$;	$-m_{z} = 0.03636$;	$-m_{\alpha} = 0.2993$

Tailplane

$\ell_{\ddot{z}} = 0.7854$;	$\ell_{\ddot{\alpha}} = 0.3927$;	$\ell_{\ddot{\xi}} = 0.0197$
$\ell_{\dot{z}} = 1.806$;	$\ell_{\dot{\alpha}} = 0.7112$;	$\ell_{\dot{\xi}} = -0.7958$
$\ell_{z} = 0.0918$;	$\ell_{\alpha} = 1.273$;	$\ell_{\xi} = 0.9126$
$-m_{\ddot{z}} = 0.3927$;	$-m_{\ddot{\alpha}} = 0.2209$;	$-m_{\ddot{\xi}} = 0.0141$
$-m_{\dot{z}} = 0.6021$;	$-m_{\dot{\alpha}} = 0.5705$;	$-m_{\dot{\xi}} = -0.00369$
$-m_{z} = 0.02296$;	$-m_{\alpha} = 0.3183$;	$-m_{\xi} = 0.3751$
$-h_{\ddot{z}} = 0.01969$;	$-h_{\ddot{\alpha}} = 0.01412$;	$-h_{\ddot{\xi}} = 0.00197$
$-h_{\dot{z}} = 0.01581$;	$-h_{\dot{\alpha}} = 0.02939$;	$-h_{\dot{\xi}} = 0.01278$
$-h_{z} = 0.000603$;	$-h_{\alpha} = 0.008357$;	$-h_{\xi} = 0.00894$

The values of the various coefficients may now be determined and are given below.

Inertia coefficients

$$a_{11} = 0.1427 + 0.797 \times 10^{-6} M$$

$$a_{12} = 0.0059214 - 15.94 \times 10^{-6} M$$

$$a_{22} = 0.007971 + 318.8 \times 10^{-6} M$$

where M is in lb.

---

[1] H.A. Jahn. "Comparison of the Experimental Wing-Aileron Derivatives of R & M. 1155 with Two-Dimensional Vortex Sheet Derivatives." R.A.E. Tech. Note No. S.M.E. 276.

The term in M in a_{11} (and similarly for a_{12}) is obtained from $\frac{1}{2\rho c_m^4 s_w} \ell^2 f_3''^2$ (where $\ell f_3''$ is the fuselage displacement at the balance weight) and not from $\frac{1}{2\rho c_m^4 s_w} (\ell f_3' - r F_2')^2$ as quoted earlier.

This is a usual procedure as some simplification is effected in certain cases and the error involved is negligibly small, since it is a function of the fuselage curvature between the elevator and mass balance hinges.

Stiffness coefficient

$$e_{11} = \frac{33.5534 \times 10^3}{V^2}$$

It should be noted that the value for e_{11} is that appropriate to zero mass balance weight, and is assumed to remain constant with variation in M. In point of fact variation of M would produce some change in mode and frequency but since these are assumed to remain constant the same assumption is applied to e_{11}.

Aerodynamic coefficients

$\gamma_{11} = 0.005041$;	$b_{11} = 0.013735$;	$c_{11} = 0.00567$
$\gamma_{12} = 0.000295$;	$b_{12} = -0.01264$;	$c_{12} = 0.02993$
$\gamma_{21} = 0.000295$;	$b_{21} = 0.000584$;	$c_{21} = 0.000167$
$\gamma_{22} = 0.000113$;	$b_{22} = 0.00117$;	$c_{22} = 0.00131$

As in Section 4 the values of the functions p_o to p_4 are now obtained as follows:-

$$p_o = 0.0011556 + 46.908 \times 10^{-6} M$$

$$p_1 = 0.0003588 + 4.1875 \times 10^{-6} M$$

$$p_2 = 0.00006834 + 2.2883 \times 10^{-6} M + 0.008084\, e_{11} + 318.8 \times 10^{-6} M\, e_{11}$$

$$p_3 = 0.000009261 + 0.00117\, e_{11}$$

$$p_4 = 0.000002429 + 0.00131\, e_{11}$$

The eliminant $p_1 p_2 p_3 - p_o p_3^2 - p_1^2 p_4 = 0$ reduces to the following:-

$$(1.5619\, M^2 e_{11}^2 + 109.2346\, M e_{11}^2 + 1811.82\, e_{11}^2)$$
$$+ (0.006035\, M^2 e_{11} - 2.2659\, M e_{11} - 138.163\, e_{11})$$
$$+ (0.0000461\, M^2 - 0.001072\, M - 0.0001852) = 0$$

For a given value of M this equation reduces to a quadratic in e_{11} from which the value of V may be determined.

The corresponding value of the frequency parameter is derived from the equation

$$\omega_m^2 = \frac{p_3}{p_1}$$

In Fig.3 a curve is shown of flutter speed plotted against mass balance weight, from which it is apparent that the speed increases with increase of weight. Values of the frequency parameter have been determined for various values of the balance weight, and it may be seen that the frequency parameter decreases as the weight increases. The deviation of ω_m from the assumed value of 0.5 is quite large for values of M greater than 30 lb. However, a value for the balance weight of 25 lb would give static balance of the elevator and this value is not likely to be greatly exceeded in any practical case. Therefore the assumed and final values of ω_m are in sufficiently good agreement within the practical range of M for it to be unnecessary to revise the initial assumed value of 0.5.

It is a usual practice to allow a safety margin of about 20% on theoretical flutter speeds, and on this basis, with the foregoing assumptions, this particular aircraft could be cleared to about 450 knots with a statically balanced elevator. This, of course, omits consideration of the effect of compressibility, and in fact for aircraft flying at speeds where compressibility effects are pronounced the permissible speed should be reduced.

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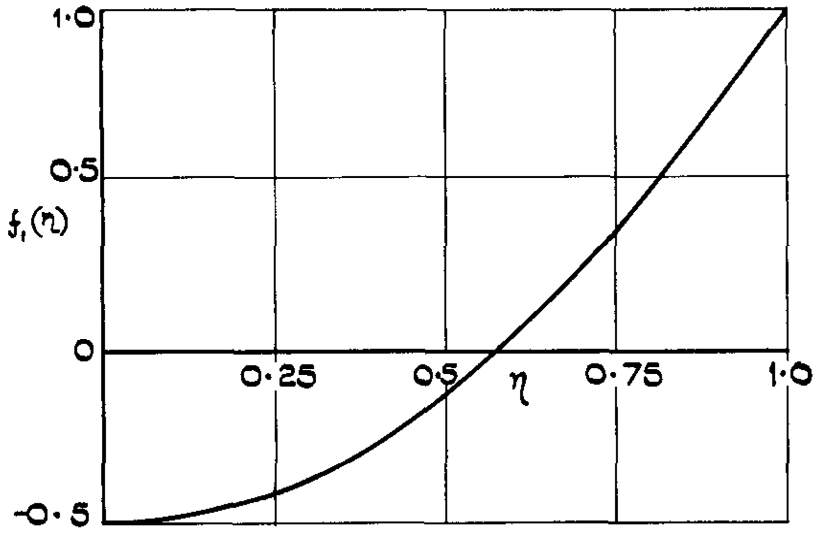
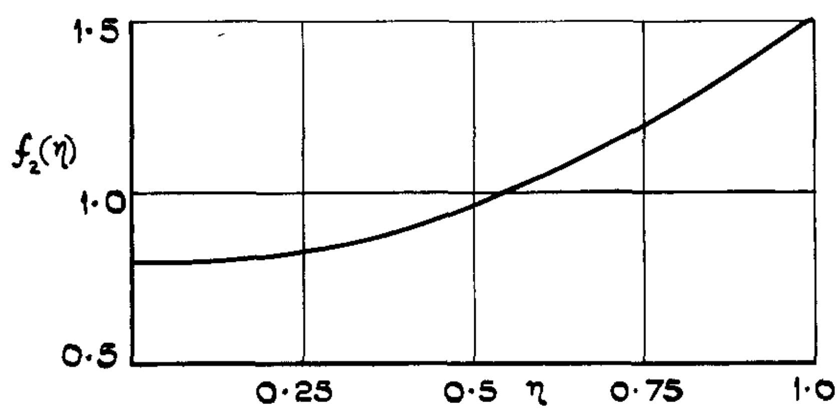
APPENDIX III

FIG. 1.



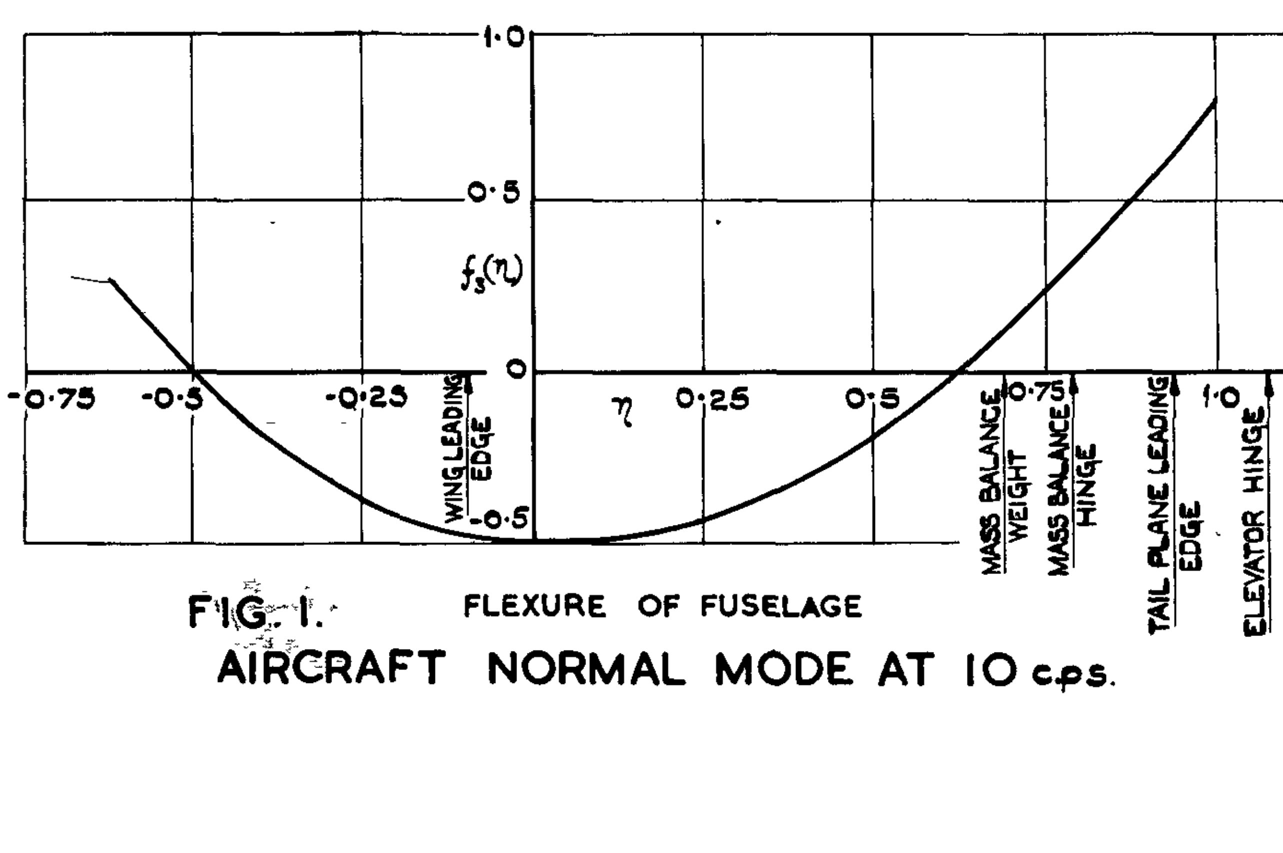
FLEXURE OF WING 1/4 CHORD LINE

ZERO WING TORSION



FLEXURE OF TAILPLANE 1/4 CHORD LINE

UNIFORM SPANWISE TAILPLANE INCIDENCE, $F_2(\eta) = 0{\cdot}128$ RADS



FLEXURE OF FUSELAGE

FIG. 1.

AIRCRAFT NORMAL MODE AT 10 c.p.s.

APPENDIX III

FIG. 2.

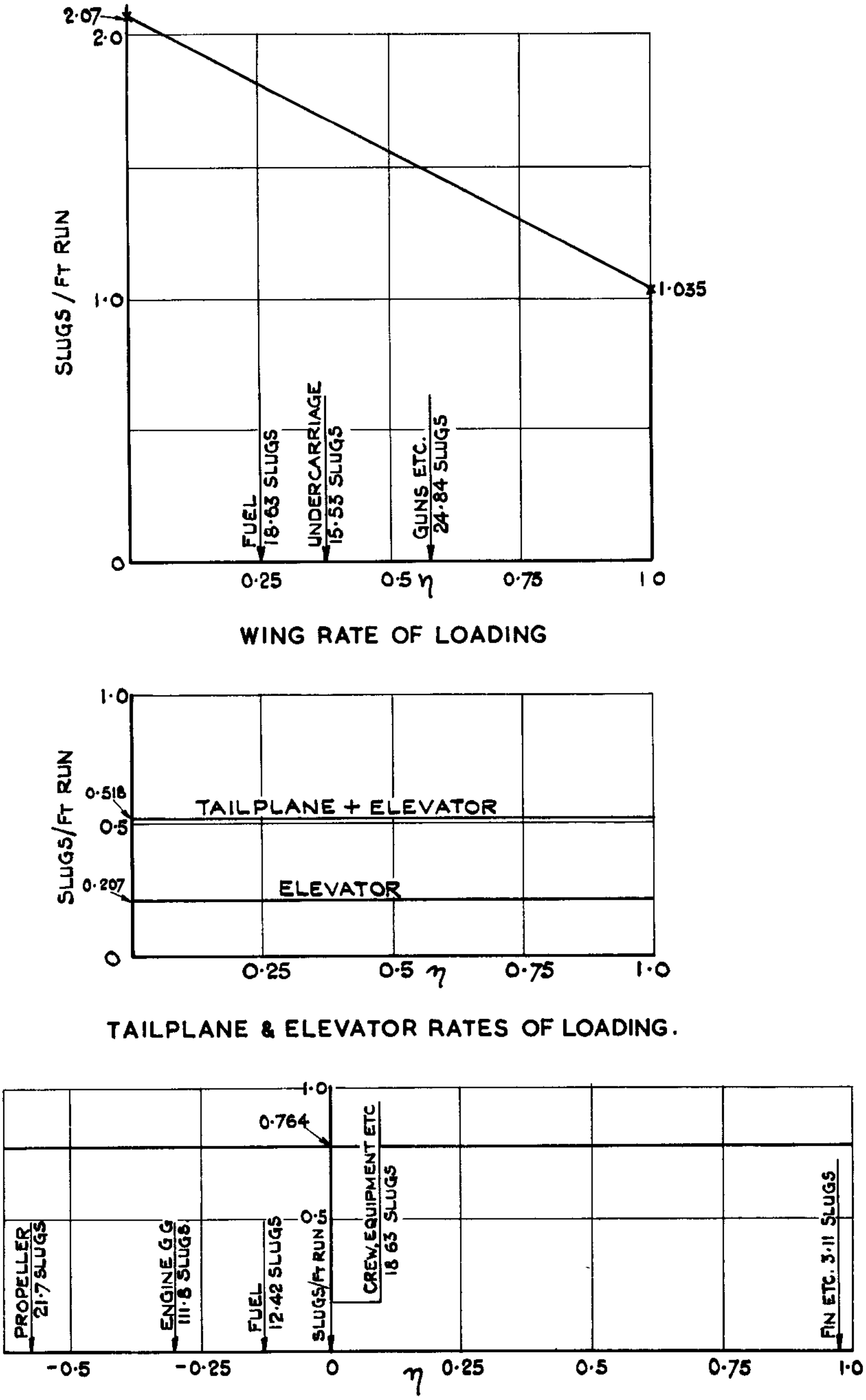


FIG. 2. LOADING DIAGRAM

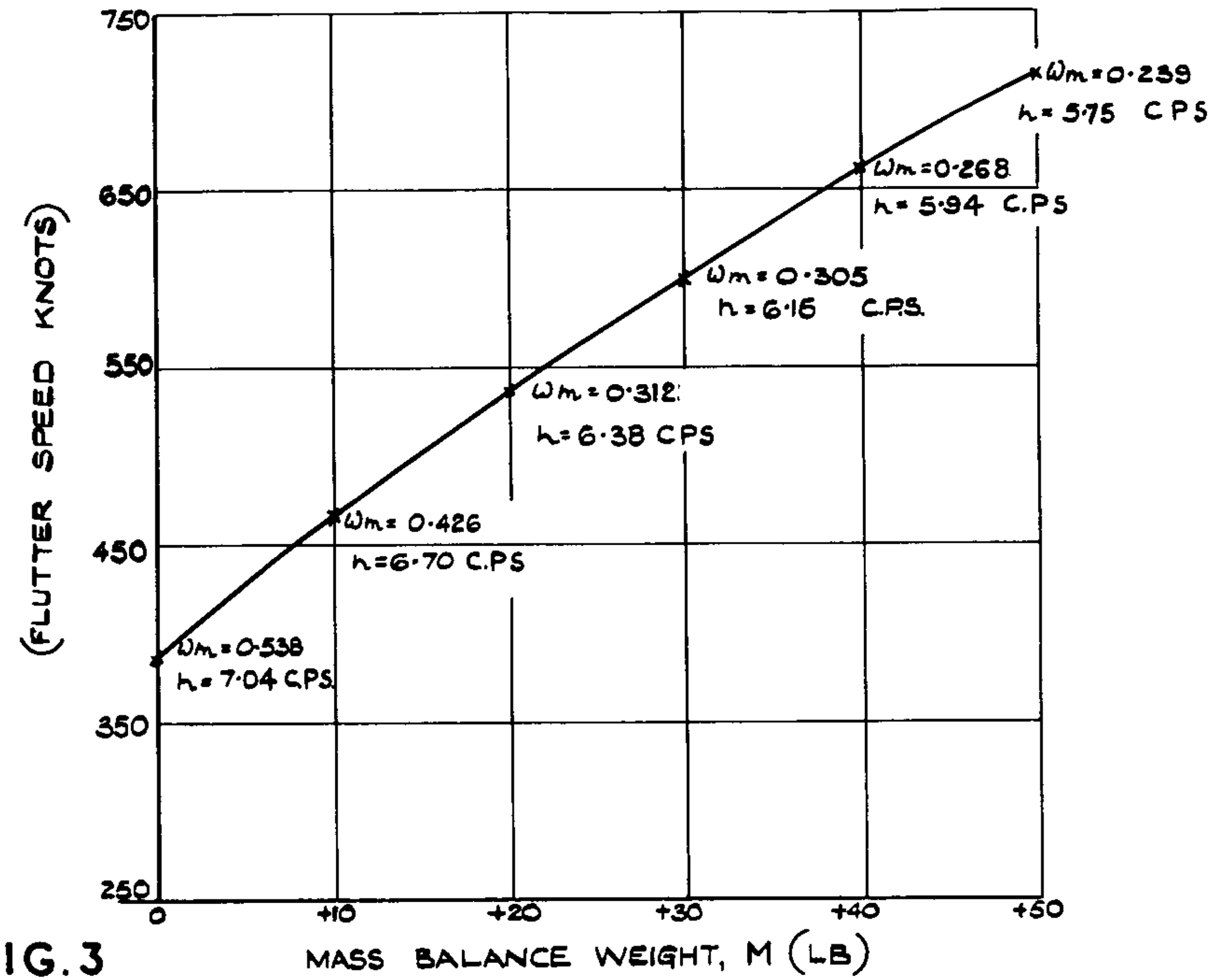


FIG. 3

VARIATION OF FLUTTER SPEED WITH ELEVATOR MASS BALANCE WEIGHT.

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