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By

F/Lt. L. C. Woods, M.A., D.Phil., A.F.R.Ae.S.,
(R N.Z.A.F. Scientific Defence Corps,
seconded to the Aerodynamics Division of the N.P L.)

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Summary

Recently published methods[1] of deducing practical values of the various control characteristics from a knowledge of their theoretical values increases the importance of the theory of two-dimensional controls in an inviscid compressible fluid. The classical work of Glauert[2] neglects compressibility and aerofoil thickness, and while the more recent work of Goldstein and Preston[3] includes thickness effects it ignores compressibility. Furthermore this latter method achieves accuracy for thick aerofoils at the cost of a complicated method of calculation.

This paper presents a theory of two-dimensional controls in compressible flow which is almost as simple to apply as Glauert's theory and is as accurate as the method of Ref. 3. An example given by Goldstein and Preston is treated by the author's method to illustrate this point.

1. Introduction

Definition of Symbols

(x,y)	the physical plane of zero incidence, with an Argand plane
z	$= x + iy,\ i = \sqrt{-1}$
(n,s)	distances measured normal to and along a streamline respectively
(q,θ)	velocity vector in polar co-ordinates
α	absolute angle of incidence, i.e., measured from the no-lift position
η	flap deflection, measured positively for a downward movement of the flap
α,η	as suffixes to denote values at absolute incidences of α and flap deflections of η
∞	as a suffix to denote values at an infinite distance from the aerofoil
U	$= q_\infty$
γ	ratio of specific heats

ρ, ρ_0/

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ρ, ρ_0 local and stagnation densities respectively

(ϕ, ψ) plane of velocity equipotentials (ϕ = constant) and streamlines (ψ = constant) for zero circulation (α = 0), where

$$d\phi = q\,ds\,, \quad d\psi = \frac{\rho}{\rho_0}\,q\,dn \qquad \ldots (1)$$

M local Mach number

$\beta \equiv (1 - M^2)^{\frac{1}{2}}$

m is defined by the equation

$$m = (1 - M^2)^{\frac{1}{2}}\,\frac{\rho_0}{\rho} = \beta\,\frac{\rho_0}{\rho} \qquad \ldots (2)$$

r is defined by the equation

$$r = \int_{q=U}^{q} \tfrac{1}{2}(m + m_\infty)\,\frac{\rho}{\rho_0}\,d\left(\log\frac{U}{q}\right) = r\left(\frac{q}{U}\right) \qquad \ldots (3)$$

w is defined by the equation

$$w = \phi + i m_\infty \psi \qquad \ldots (4)$$

z_α the physical plane for an absolute incidence of α, i.e.,

$$z_\alpha = e^{i\alpha} z \qquad \ldots (5)$$

(δ, γ) elliptic co-ordinates defined by

$$w = -2a\cosh\zeta = -2a\cosh(\delta + i\gamma); \qquad \ldots (6)$$

the aerofoil surface is $\psi = 0$, $-2a \leqslant \phi \leqslant 2a$, or $\delta = 0$, when

$$\phi = -2a\cos\gamma \qquad \ldots (7)$$

c aerofoil chord when $\eta = 0$

$(1 - E)c$ the contour of the flap when undeflected meets the upper and lower surfaces of the aerofoil at $x = (1 - E)c$, thus Ec is the "flap chord"

$(1 - E')c$ distance of hinge from leading edge of the aerofoil ($E \doteqdot E'$)

α'/

α' incidence of the front part of the aerofoil measured from the $\eta = 0$ chord line

$-\alpha_o, -\alpha_o'$ no-lift angles for $\eta = 0$ and $\eta \neq 0$ respectively, thus

$$\alpha = \alpha' + \alpha_o , \ \eta = 0 \qquad \dots (8)$$

and

$$\alpha = \alpha' + \alpha_o' , \ \eta \neq 0 \qquad \dots (9)$$

C_p pressure coefficient

C_L lift coefficient

C_m moment coefficient

C_H hinge moment coefficient, such that the hinge moment is

$$\tfrac{1}{2} \rho_\infty U^2 E^2 c^2 \ C_H$$

a_o, a_1, a_2 $\quad a_o = (C_L)_{\alpha'=\eta=0} , \ a_1 = \left(\frac{\partial C_L}{\partial \alpha}\right)_{\alpha'=\eta=0} , \ a_2 = \left(\frac{\partial C_L}{\partial \eta}\right)_{\alpha'=\eta=0} ,$

whence to first order in α' and η

$$C_L = a_o + a_1 \alpha' + a_2 \eta \qquad \dots (10)$$

h, m_o $\quad h = -\left(\frac{\partial C_m}{\partial C_L}\right)_{C_L=\eta=0} , \ m_o = -\left(\frac{\partial C_m}{\partial \eta}\right)_{C_L=\eta=0} ,$

i.e., to first order

$$C_m = -h \, C_L - m_o \, \eta \qquad \dots (11)$$

b_o, b_1, b_2 $\quad b_o = (C_H)_{\alpha'=\eta=0} , \ b_1 = \left(\frac{\partial C_H}{\partial \alpha}\right)_{\alpha'=\eta=0} , \ b_2 = \left(\frac{\partial C_H}{\partial \eta}\right)_{\alpha'=\eta=0}$

i.e., to first order

$$C_H = b_o + b_1 \alpha' + b_2 \eta \qquad \dots (12)$$

b $\quad b = -\left(\frac{\partial C_H}{\partial \eta}\right)_{C_L=\eta=0} .$

With/

With the aid of (10) equation (12) can be written

$$C_H = b_o - \frac{b_1 a_o}{a_1} + \frac{b_1}{a_1} C_L - \left(\frac{b_1 a_2 - a_1 b_2}{a_1}\right)\eta,$$

so that

$$b = \frac{b_1 a_2 - a_1 b_2}{a_1} . \qquad ...(13)$$

This paper gives methods of calculating the quantities a_o, a_1, a_2, m_o, h, b_o, b_1, b_2 and b defined above, in subsonic two-dimensional flow. Compressibility effects on these parameters are calculated by a theory more accurate than linear perturbation theory, but not valid above the critical Mach number. The theory is applicable to aerofoils of moderate thickness (say up to 20% thick) and for small values of η.

An exact method for the calculation of a_o, a_1 and h for aerofoils of any thickness in incompressible flow is given in Appendix I. The exact theory of the hinged flat plate in incompressible flow but without restrictions on the value of η is given in Appendix IV. A summary of formulae is given in Section 4.

The independent variables of the theory to be given in the next section are δ and γ defined by equations (4) and (6), while the dependent variables are r (equation(3)) and θ . The quantity r can be readily evaluated as a function of q/U . It has been shown (see Ref. 5) that when the approximation

$$m = m_\infty \qquad ...(14)$$

is admissible, r and θ are conjugate harmonic functions in the w plane. (The theory is outlined in Appendix V for the reader's convenience.) Equation (14) and an equation similar to (3) were first used by von Kármán[6] to show that ϕ and ψ are approximately harmonic functions in the (r,θ) plane. Although the theory given below is not really valid when M_∞ is greater than that critical value corresponding to the first appearance of sonic speed locally (c.f. equation (2)), it can be still applied with some confidence to calculate the subsonic field when small supersonic patches exist. This point is important in the theory of controls as a high but localized velocity peak does occur at the flap hinge on the upper surface when η is positive.

The complex number defined by

$$f = r + i\theta \qquad ...(15)$$

is approximately an analytic function of w (r and θ being conjugate harmonic functions), but if the flow is incompressible, $r = \log(U/q)$, $w = \phi + i\psi$, and so

$$f = \log\left(\frac{U}{q} e^{i\theta}\right) = \log\left(\frac{Udz}{dw}\right) , \qquad ...(16)$$

whence/

whence it follows that f is exactly an analytic function of w. Thus the theory of Section 2 (but not of Section 3) will be exact in incompressible flow.

2. Basic Mathematical Theory

The theory of this section is quite general and applies to aerofoils with or without deflected flaps.

If θ and θ_α are measured from the direction of flow at infinity, i.e., if $\theta_\infty = \theta_{\alpha\infty} = 0$, it follows from equations (3) and (15) that

$$f_\infty = f_{\alpha\infty} = 0 . \qquad \ldots(17)$$

Now f is an analytic function of w and therefore (see equation (6)) it is an analytic function of ζ. In fact, as shown in Ref. 4,*

$$f(\zeta) = -\frac{1}{\pi}\int_{\gamma^*=-\pi}^{\pi} \log \sinh \tfrac{1}{2}(i\gamma^* - \zeta)\, d\theta(\gamma^*) , \qquad \ldots(18)$$

where $\theta(\gamma^*)$ is the value of θ on the aerofoil surface. Equation (18) is the no-lift solution. If the aerofoil is placed at a small absolute angle of incidence α, then on the Joukowski Hypothesis, as in Ref. 5,*

$$f_\alpha(\zeta) = f(\zeta) - i\alpha - \log \frac{\sinh \tfrac{1}{2}(\zeta + 2i\alpha)}{\sinh \tfrac{1}{2}\zeta} , \qquad \ldots(19)$$

in which it is assumed that the trailing edge is at $\gamma = \pi$, and the stream direction is from $x = -\infty$ (see Fig. 1). The form of equation (19) shows that the effect of incidence on the front stagnation point is to displace it from $\gamma = 0$ to $\gamma = -2\alpha$.

Important auxiliary equations can be deduced by considering the form f_α takes near infinity. From equations (18) and (19) it follows that

$$f = +\frac{1}{\pi}\int_{\gamma^*=-\pi}^{\pi} (\tfrac{1}{2}\zeta + \log 2)\, d\theta(\gamma^*) - \frac{i}{2\pi}\int_{\gamma^*=-\pi}^{\pi} \gamma^*\, d\theta(\gamma^*)$$

$$+ e^{+\zeta}\left\{2ie^{+i\alpha}\sin\alpha + \frac{1}{\pi}\int_{\gamma^*=-\pi}^{\pi} e^{-i\gamma^*}\, d\theta(\gamma^*)\right\}$$

$$+ e^{+2\zeta}\left\{ie^{+2i\alpha}\sin 2\alpha + \frac{1}{2\pi}\int_{\gamma^*=-\pi}^{\pi} e^{-2i\gamma^*}\, d\theta(\gamma^*)\right\} + O(e^{+3\zeta}) .$$

*See also Appendix V.

Comparing/

Comparing this with equation (17) we conclude that

$$\int_{y^*=-\pi}^{\pi} d\theta(y^*) = 0, \qquad \ldots(20)$$

and

$$\int_{y^*=-\pi}^{\pi} y^* \, d\theta(y^*) = - \int_{-\pi}^{\pi} \theta(y^*) \, dy^* = 0. \qquad \ldots(21)$$

Equation (20) is the obvious requirement that $\theta(y^*) = \theta(2\pi + y^*)$, while equation (21) fixes the orientation of the aerofoil for the no-lift position. If $\underset{\sim}{\theta}$ is measured from the aerofoil chord then $\theta = \underset{\sim}{\theta} + \alpha_0$, and (21) yields

$$\alpha_0 = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \underset{\sim}{\theta}(y^*) \, dy^*, \qquad \ldots(22)$$

which fixes the value of the no-lift angle.

From equation (6), $w \to \infty$ implies that $\zeta \to -\infty$, and we find

$$e^{+\zeta} = -\frac{a}{w} + O\left(\frac{a}{w}\right)^3,$$

and so the expansion for f_α can be written

$$f_\alpha = -\frac{a}{w}\left\{2ie^{+i\alpha}\sin\alpha + \frac{1}{\pi}\int_{y^*=-\pi}^{\pi} e^{-iy^*} \, d\theta(y^*)\right\}$$

$$+ \left(\frac{a}{w}\right)^2\left\{ie^{+2i\alpha}\sin 2\alpha + \frac{1}{2\pi}\int_{y^*=-\pi}^{\pi} e^{-2iy^*} \, d\theta(y^*)\right\} + O\left(\frac{a}{w}\right)^3. \qquad \ldots(23)$$

From this equation we conclude that

$$\int_{y^*=-\pi}^{\pi} \cos y^* \, d\theta(y^*) = \int_{y^*=-\pi}^{\pi} \sin y^* \, d\theta(y^*) = 0, \qquad \ldots(24)$$

otherwise when $\alpha = 0$, f will have a term $O\left(\frac{a}{w}\right)$, and since from equation (3)

$$\frac{q}{U} \doteqdot e^{-r/\beta_\infty}, \qquad \ldots(25)$$

q/U will be of the form $1 + A/|w|$ for large $|w|$, and a lift-producing circulation will exist. (An alternative proof for the case of incompressible flow appears in Appendix I.)

Finally it follows from equations (1) and (7) that on the aerofoil surface

$$\frac{sU}{2a} = \int_0^{\gamma} \frac{U \sin \gamma}{q}\, d\gamma , \qquad \ldots(26)$$

where the origin of s is taken at the front stagnation point.

This completes the account of the basic mathematical theory. The numerical application of this theory to the calculation of the compressible flow about aerofoils is given in Ref. 5.

3. The Aerofoil with a Hinged Flap at Small Angles of Deflection

The theory to be given below is only valid for small values of η, the flap deflection angle, Unfortunately a simple theory valid for large values of η (say $> 20^\circ$) is not possible, except in the case of a hinged flat plate (Appendix IV). In general if η is large the only recourse is to find the flow about the aerofoil and flap ab initio for each value of η. The author's polygon method[5] described in the previous section, would be very suitable for such a calculation. However, as shown below, a relatively simple theory applicable even to comparatively thick aerofoils can be developed when terms $O(\eta^2)$ can be neglected.

3.1 The Velocity Distribution

Subscripts α and η will be used to denote values when the aerofoil is at an incidence absolute α with a flap deflection η, while the absence of subscripts denote the case $\alpha = \eta = 0$. Consider the aerofoil, for which $\alpha = \eta = 0$, shown in Fig. 2(a). We shall suppose that the solution has been obtained for this case, and that therefore we have or can deduce q/U and s/c as functions of γ (defined by equation (7) and in Fig. 1). If the polygon method has been used to find the solution, q/U and s/c will be immediately available as functions of γ (see example (b) in Section 5); otherwise suppose q/U is given as a function of s, then the equation

$$\frac{\phi}{2a} = -\cos\gamma = \left(\frac{cU}{2a}\right) \int_0^{s/c} \frac{q}{U}\, d\left(\frac{s}{c}\right) - 1, \qquad \ldots(27)$$

which follows from (1) and (7), enables $s/c = s/c(\gamma)$, and hence $q/U = q/U(\gamma)$ to be calculated. The constant $(cU/2a)$ must satisfy

$$1 \doteqdot \left(\frac{cU}{2a}\right) \int_0^{p/c} \frac{q}{U}\, d\left(\frac{s}{c}\right),$$

where p is the perimeter distance from the leading to the trailing edge.

In/

In Fig. 2 the flap surface is shown starting at C, where $\gamma = \lambda_o$, and F, where $\gamma = -\lambda_1$. When $\eta = 0$, each of C and F correspond to a value of x/c of $1 - E$. The hinge will be taken to be at $x/c = 1 - E'$, and of course for thin aerofoils $E \doteq E'$.

The most important increments (θ_p, say) to θ due to the deflection of the flap are shown in Fig. 3. They are due to (i) the front stagnation point shifts to some point B, where $\gamma = \lambda$ say, and consequently the flow direction between A and B is reversed, i.e., θ is decreased by π in $0 \leqslant \gamma \leqslant \lambda$, (ii) the deflection of the flap reduces θ by η in $-\pi \leqslant \gamma \leqslant -\lambda_1$, $\lambda_o \leqslant \gamma \leqslant \pi$, and (iii) θ is increased by $\alpha_o' - \alpha_o$ in $-\pi \leqslant \gamma \leqslant \pi$ due to a change in the no-lift angle from α_o to α_o'. Unfortunately these are not the only increments to θ, for the modification to the velocity distribution which they produce (equation (39) below) slightly distorts the relation between s and γ (equation (26)) and consequently causes a slight change ($\Delta\theta$) in $\theta(\gamma)$. We can thus write θ for $\eta \neq 0$ as

$$\theta_\eta = \theta_o + \theta_p + \Delta\theta ,$$

where θ_o is the value of θ when $\eta = 0$. For a thin aerofoil the distortion in the (s, γ) relation will result in quite small values of $\Delta\theta$ away from the nose of the aerofoil as $\Delta\theta = \Delta s/R$, where Δs is the change in s. The largest values of $\Delta\theta$ will be near the nose, but these will have a comparatively small effect on the velocity distribution over the flap, and therefore on C_H. Thus only a small error will be introduced (except in the velocity distribution near the nose) by writing

$$\theta_\eta = \theta_o + \theta_p . \qquad \ldots(28)$$

Now θ_o satisfies equations (20), (21) and (24), and since θ_η must also satisfy these equations, this must also be true of θ_p. The increment θ_p is a step function with jumps in value as set out in the following table:-

γ	$-\pi$	$-\lambda_1$	0	λ	λ_o	π
Jump in θ_p	$-\eta + \alpha_o' - \alpha_o$	η	$-\pi$	π	$-\eta$	$\eta - \alpha_o' + \alpha_o$

and consequently the Stieltjes integrals in equations (21) and (24) degenerate to

$$2\pi(\eta - \alpha_o' + \alpha_o) - \eta(\lambda_1 + \lambda_o) + \pi\lambda = 0 \qquad \ldots(29)$$

$$\eta(\cos\lambda_o - \cos\lambda_1) + \pi(1 - \cos\lambda) = 0 \qquad \ldots(30)$$

$$\eta(\sin\lambda_o + \sin\lambda_1) - \pi\sin\lambda = 0 . \qquad \ldots(31)$$

Equation/

Equation (20) is obviously satisfied by θ_p. Equations (30) and (31) yield

$$\lambda_o - \lambda_1 = \lambda \qquad \ldots(32)$$

and

$$\sin \tfrac{1}{2}\lambda = \frac{\eta}{\pi} \sin \lambda_m \qquad \ldots(33)$$

where

$$\lambda_m = \tfrac{1}{2}(\lambda_o + \lambda_1) . \qquad \ldots(34)$$

These equations imply that we cannot fix the positions of C and F (Fig. 2) independently. It is convenient to regard η and λ_m as the dependent variables. Equation (29) fixes the value of $(\alpha_o' - \alpha_o)$, the change in no-lift angle due to the flap deflection. Using equation (33) and ignoring terms $O(\eta^2)$ we find

$$\alpha_o' - \alpha_o = \eta\left(1 - \frac{\lambda_m}{\pi} + \frac{\sin \lambda_m}{\pi}\right), \qquad \ldots(35)$$

whence

$$\left(\frac{\partial(\alpha_o' - \alpha_o)}{\partial \eta}\right)_{\eta=0} = 1 - \frac{\lambda_m}{\pi} + \frac{\sin \lambda_m}{\pi} . \qquad \ldots(36)$$

In Appendix III it is shown that these equations are exact for incompressible flow about a flat hinged plate.

Substitution of equation (28) in equation (18) yields

$$f_{o,\eta}(\zeta) = f(\zeta) - i(\eta - \alpha_o' + \alpha_o) + \frac{\eta}{\pi} \log \frac{\sinh \tfrac{1}{2}(\zeta - i\lambda_o)}{\sinh \tfrac{1}{2}(\zeta + i\lambda_1)} + \log \frac{\sinh \tfrac{1}{2}\zeta}{\sinh \tfrac{1}{2}(\zeta - i\lambda)} .$$

If the aerofoil is now placed at an absolute incidence of α the front stagnation point will be displaced from $\gamma = \lambda$ to $\gamma = \lambda - 2\alpha$, and hence (c.f. equation (19)) we will have

$$f_{\alpha,\eta}(\zeta) = f(\zeta) - i(\eta - \alpha_o' + \alpha_o + \alpha) + \frac{\eta}{\pi} \log \frac{\sinh \tfrac{1}{2}(\zeta - i\lambda_o)}{\sinh \tfrac{1}{2}(\zeta + i\lambda_1)}$$

$$+ \log \frac{\sinh \tfrac{1}{2}\zeta}{\sinh \tfrac{1}{2}(\zeta + 2i\alpha - i\lambda)} . \qquad \ldots(37)$$

On the aerofoil surface, $\delta = 0$, and equation (37) becomes, with the aid of (32) and (34)

$$r_{\alpha,\eta}(\gamma) = r(\gamma) + \frac{\eta}{\pi} \log \frac{\sin \tfrac{1}{2}(\gamma - \tfrac{1}{2}\lambda - \lambda_m)}{\sin \tfrac{1}{2}(\gamma - \tfrac{1}{2}\lambda + \lambda_m)} + \log \frac{\sin \tfrac{1}{2}\gamma}{\sin \tfrac{1}{2}(\gamma + 2\alpha - \lambda)} , \qquad \ldots(38)$$

where/

where λ and λ_m are related by equation (33). The velocity distribution now follows from equation (3). At low Mach numbers the approximation (25) is valid, when equation (38) yields

$$\frac{q_{\alpha,\eta}(\gamma)}{U} = \frac{q}{U}\left\{\frac{\sin\frac{1}{2}(\gamma - \frac{1}{2}\lambda + \lambda_m)}{\sin\frac{1}{2}(\gamma - \frac{1}{2}\lambda - \lambda_m)}\right\}^{\eta/\pi\beta_\infty}\left\{\frac{\sin\frac{1}{2}(\gamma + 2\alpha - \lambda)}{\sin\frac{1}{2}\gamma}\right\}^{1/\beta_\infty} . \qquad \ldots(39)$$

In the calculation of the various derivatives appearing in equations (10), (11) and (12) it will be convenient at first to regard α and η as independent variables. Subsequently α will be replaced by (equations (9) and (35))

$$\alpha = \alpha' + \alpha_0 + \eta\left(1 - \frac{\lambda_m}{\pi} + \frac{\sin\lambda_m}{\pi}\right), \qquad \ldots(40)$$

so that α' and η become the independent variables.

3.2 Calculation of C_L, a_0, a_1, a_2

The lift coefficient, C_L, is defined by the contour integral taken round the aerofoil surface

$$C_L = -\frac{1}{c}\oint C_p \cos\theta \, ds ,$$

where the pressure coefficient C_p is a function of γ, η and α. Thus, since

$$\frac{1}{c}\cos\theta \, ds = \frac{\cos\theta}{c\,q}\, d\phi = \left(\frac{2a}{Uc}\right)\left(\frac{U\cos\theta}{q}\right)\sin\gamma \, d\gamma ,$$

$$C_L = -\left(\frac{2a}{Uc}\right)\int_{-\pi}^{\pi} C_p \sin\gamma\left(\frac{U\cos\theta}{q}\right)d\gamma . \qquad \ldots(41)$$

If υ is the ratio of the specific heats, C_p is given by

$$C_p = \frac{2}{\upsilon M_\infty^2}\left\{\left[1 - \frac{\upsilon - 1}{2} M_\infty^2\left\{\left(\frac{q_{\alpha,\eta}}{U}\right)^2 - 1\right\}\right]^{\upsilon/\upsilon-1} - 1\right\},$$

from which it follows that

$$\frac{\partial C_p}{\partial(q/U)} = -2\left(\frac{q}{U}\right)\frac{\rho}{\rho_\infty} . \qquad \ldots(42)$$

It/

It is easily deduced from equations (3), (33) and (38) that

$$\left(\frac{\partial(q/U)}{\partial\alpha}\right)_{\alpha=\eta=0} = \frac{1}{\beta_\infty}\left(\frac{q}{U}\right)\left(\frac{2m_\infty}{m+m_\infty}\right)\frac{\rho_\infty}{\rho}\cot\tfrac{1}{2}\gamma\,,$$

and

$$\left(\frac{\partial(q/U)}{\partial\eta}\right)_{\alpha=\eta=0} = -\frac{1}{\beta_\infty}\left(\frac{q}{U}\right)\left(\frac{2m_\infty}{m+m_\infty}\right)\frac{\rho_\infty}{\rho}\left\{\frac{1}{\pi}\log\frac{\sin\frac{1}{2}(\gamma-\lambda_m)}{\sin\frac{1}{2}(\gamma+\lambda_m)} + \frac{\sin\lambda_m}{\pi}\cot\tfrac{1}{2}\gamma\right\},$$

and hence from equation (42)

$$\left(\frac{\partial C_p}{\partial\alpha}\right)_{\alpha=\eta=0} = -\frac{2}{\beta_\infty}\chi\left(\frac{q}{U}\right)^2\cot\tfrac{1}{2}\gamma\,, \qquad \ldots(43)$$

and

$$\left(\frac{\partial C_p}{\partial\eta}\right)_{\alpha=\eta=0} = \frac{2}{\beta_\infty}\chi\left(\frac{q}{U}\right)^2\left\{\frac{1}{\pi}\log\frac{\sin\frac{1}{2}(\gamma-\lambda_m)}{\sin\frac{1}{2}(\gamma+\lambda_m)} + \frac{\sin\lambda_m}{\pi}\cot\tfrac{1}{2}\gamma\right\}, \qquad \ldots(44)$$

where

$$\chi = \frac{2m_\infty}{m+m_\infty}\,,$$

is a function of q/U. This function is given in Table 2 of Ref. 5 for M_∞ = 0.5, 0.7 and 0.79. Differentiating equation (41) with respect to α and η, and making use of equations (43) and (44), we find

$$\left(\frac{\partial C_L}{\partial\alpha}\right)_{\alpha=\eta=0} = \frac{2}{\beta_\infty}\left(\frac{4a}{Uc}\right)\int_{-\pi}^{\pi}\chi\left(\frac{q}{U}\cos\theta\right)\cos^2\tfrac{1}{2}\gamma\,d\gamma\,, \qquad \ldots(45)$$

and

$$\left(\frac{\partial C_L}{\partial\eta}\right)_{\alpha=\eta=0} = -\frac{1}{\beta_\infty}\left(\frac{4a}{Uc}\right)\int_{-\pi}^{\pi}\chi\left(\frac{q}{U}\cos\theta\right)\sin\gamma$$

$$\times\left\{\frac{1}{\pi}\log\frac{\sin\frac{1}{2}(\gamma-\lambda_m)}{\sin\frac{1}{2}(\gamma+\lambda_m)} + \frac{\sin\lambda_m}{\pi}\cot\tfrac{1}{2}\gamma\right\}d\gamma\,. \qquad \ldots(46)$$

If the polygon method of calculating q/U has been used, $(4a/Uc)$, $\frac{q}{U}(\gamma)$ and $\theta(\gamma)$ will be known, $\chi(\gamma)$ can be readily deduced from tables such as those given in Ref. 5, and so the integral in (45) can be evaluated numerically without difficulty. A calculation of this type appears in Ref. 5.

A/

A simple approximation can be found by writing

$$\chi = \frac{q}{U} \cos\theta = 1 \qquad \ldots(47)$$

in the integrals of equations (45) and (46). We find

$$\left(\frac{\partial C_L}{\partial \alpha}\right)_{\alpha=\eta=0} = \frac{2\pi}{\beta_\infty}\left(\frac{4a}{Uc}\right), \qquad \ldots(48)$$

and

$$\left(\frac{\partial C_L}{\partial \eta}\right)_{\alpha=\eta=0} = 0 . \qquad \ldots(49)$$

Equation (49) is in any case obvious since C_L depends only on α. From

$$C_L = \alpha\left(\frac{\partial C_L}{\partial \alpha}\right)_{\alpha=\eta=0} + \eta\left(\frac{\partial C_L}{\partial \eta}\right)_{\alpha=\eta=0},$$

and equations (40), (48) and (49) it follows that

$$C_L = \frac{2\pi}{\beta_\infty}\left(\frac{4a}{Uc}\right)\left\{\alpha_0 + \alpha' + \eta\left(1 - \frac{\lambda_m}{\pi} + \frac{\sin\lambda_m}{\pi}\right)\right\}.$$

A comparison of this equation with equation (10) yields

$$a_0 = \frac{2\pi}{\beta_\infty}\left(\frac{4a}{Uc}\right)\alpha_0 \qquad \ldots(50)$$

$$a_1 = \frac{2\pi}{\beta_\infty}\left(\frac{4a}{Uc}\right) \qquad \ldots(51)$$

and

$$a_2/a_1 = 1 - \frac{\lambda_m}{\pi} + \frac{\sin\lambda_m}{\pi}. \qquad \ldots(52)$$

It is well-known that for thick aerofoils in incompressible flow equations (50) and (51) are exact (see Appendix I), while in Appendix IV it is shown that equation (52) is exact for the flat plate in incompressible flow. An approximation for the parameter $(4a/Uc)$, which occurs throughout the theory, is given in Appendix II.

3.3 Calculation of C_m, h and m_o

The equation corresponding to (41) for the moment coefficient about the leading edge is

$$C_m = \left(\frac{2a}{Uc}\right) \int_{-\pi}^{\pi} C_p \left(\frac{x}{c} + \frac{y}{c}\tan\theta\right) \left(\frac{U}{q}\cos\theta\right) \sin\gamma\, d\gamma,$$

where x/c is measured from the leading edge. Differentiating this equation with respect to α and η and making use of equations (43) and (44) we find

$$\left(\frac{\partial C_m}{\partial \alpha}\right)_{\alpha=\eta=0} = -\frac{2}{\beta_\infty}\left(\frac{4a}{Uc}\right) \int_{-\pi}^{\pi} \chi \left(\frac{x}{c} + \frac{y}{c}\tan\theta\right)\left(\frac{q}{U}\cos\theta\right) \cos^2 \tfrac{1}{2}\gamma\, d\gamma,$$

and

$$\left(\frac{\partial C_m}{\partial \eta}\right)_{\alpha=\eta=0} = \frac{1}{\beta_\infty}\left(\frac{4a}{Uc}\right) \int_{-\pi}^{\pi} \chi \left(\frac{x}{c} + \frac{y}{c}\tan\theta\right)\left(\frac{q}{U}\cos\theta\right)$$

$$\times \left\{\frac{1}{\pi}\log\frac{\sin\frac{1}{2}(\gamma - \lambda_m)}{\sin\frac{1}{2}(\gamma + \lambda_m)} + \frac{\sin\lambda_m}{\pi}\cot\tfrac{1}{2}\gamma\right\} \sin\gamma\, d\gamma,$$

which can be evaluated directly when q/U has been calculated by the polygon method.

Approximations to these equations can be found by writing $Ux = 2a + \phi$, which leads to

$$\frac{x}{c} = \tfrac{1}{2}\left(\frac{4a}{Uc}\right)(1 - \cos\gamma),$$

ignoring the very small " $\frac{y}{c}\tan\theta$ " term, and using equation (47). The results are

$$\left.\begin{aligned} \left(\frac{\partial C_m}{\partial \alpha}\right)_{\alpha=\eta=0} &= -\frac{\pi}{2\beta_\infty}\left(\frac{4a}{Uc}\right)^2, \\ \text{and}\quad \left(\frac{\partial C_m}{\partial \eta}\right)_{\alpha=\eta=0} &= \frac{1}{2\beta_\infty}\left(\frac{4a}{Uc}\right)^2 \sin\lambda_m(1 - \cos\lambda_m); \end{aligned}\right\} \quad \ldots(53)$$

but/

but $C_L = a_1\alpha$, and so it follows from equations (48), (53) and the definitions of h and m_0 that+

$$\left.\begin{aligned} h &= \frac{1}{4}\left(\frac{4a}{Uc}\right) \\ m_0 &= \frac{1}{2\beta_{cJ}}\left(\frac{4a}{Uc}\right)^2 \sin\lambda_m(1-\cos\lambda_m)\,. \end{aligned}\right\} \qquad \ldots(54)$$

3.4 Calculation of C_H, b_0, b_1 and b_2

By comparison with the equation for C_m given in Section 3.3 it is clear that the coefficient of the hinge moment, C_H, is given by++

$$C_H = \left(\frac{2a}{Uc}\right)\frac{1}{E^2}\left(\int_{-\pi}^{-\lambda_1} + \int_{\lambda_0}^{\pi}\right) C_p\left\{\frac{x}{c} - 1 + E' + \frac{y}{c}\tan\theta\right\}\left(\frac{U}{q}\cos\theta\right)\sin\gamma\, d\gamma\,, \qquad \ldots(55)$$

the hinge being at $x/c = 1 - E'$, where $\gamma = \lambda'_m$, say. From equations (32) and (34), $\eta \to 0$ implies $\lambda_1 \to \lambda_0 \to \lambda_m$. Thus

$$(C_H)_{\alpha=\eta=0} = \frac{1}{2}\left(\frac{4a}{Uc}\right)\frac{1}{E^2}\left(\int_{-\pi}^{-\lambda_m} + \int_{\lambda_m}^{\pi}\right) C_p\left\{\frac{x}{c} - 1 + E' + \frac{y}{c}\tan\theta\right\}\left(\frac{U}{q}\cos\theta\right)\sin\gamma\, d\gamma\,, \qquad \ldots(56)$$

which has to be calculated numerically just as in the exact treatment of equation (45)

Differentiating/

- -

+In incompressible flow this equation for h gives results accurate to within 0.01c provided the maximum thickness is less than 0.1c and occurs in the range $0.4c \leq x \leq 0.6c$. A more accurate equation for h in incompressible flow is given in Appendix III.

++Note that the "non-dimensionalizing" distance for C_H is Ec, not $E'c$.

Differentiating equation (55) we find, with the aid of equations (43) and (44), that

$$\left.\begin{aligned}\left(\frac{\partial C_H}{\partial \alpha}\right)_{\alpha=\eta=0} &= -\frac{2}{\beta_\infty E^2}\left(\frac{4a}{Uc}\right)\left(\int_{-\pi}^{-\lambda_m} + \int_{\lambda_m}^{\pi}\right)\chi\left\{\frac{x}{c} - 1 - E' + \frac{y}{c}\tan\theta\right\} \\ &\qquad \times \left(\frac{q}{U}\cos\theta\right)\cos^2\tfrac{1}{2}\gamma\, d\gamma \\ \text{and}^{*} \\ \left(\frac{\partial C_H}{\partial \eta}\right)_{\alpha=\eta=0} &= \frac{1}{\beta_\infty E^2}\left(\frac{4a}{Uc}\right)\left(\int_{-\pi}^{-\lambda_m} + \int_{\lambda_m}^{\pi}\right)\chi\left\{\frac{x}{c} - 1 - E' + \frac{y}{c}\tan\theta\right\}\left(\frac{q}{U}\cos\theta\right)\sin\gamma \\ &\qquad \times \left\{\frac{\sin\lambda_m}{\pi}\cot\tfrac{1}{2}\gamma + \frac{1}{\pi}\log\frac{\sin\frac{1}{2}(\gamma-\lambda_m)}{\sin\frac{1}{2}(\gamma+\lambda_m)}\right\}d\gamma\,.\end{aligned}\right\} \quad \ldots(57)$$

Equations (57) can be evaluated numerically, but for thin aerofoils travelling at speeds such that M_∞ is well below the critical Mach number the following approximations will be sufficiently accurate. We write

$$\chi\left\{\frac{x}{c} - 1 - E' + \frac{y}{c}\tan\theta\right\}\left(\frac{q}{U}\cos\theta\right) = \tfrac{1}{2}\left(\frac{4a}{Uc}\right)(\cos\lambda'_m - \cos\gamma),$$

which results in

$$\left(\frac{\partial C_H}{\partial \alpha}\right)_{\alpha=\eta=0} = -\frac{1}{\beta_\infty E^2}\left(\frac{4a}{Uc}\right)^2 \times \{\sin\lambda_m(1 + \tfrac{1}{2}\cos\lambda_m - \cos\lambda'_m) + (\pi - \lambda_m)(\cos\lambda'_m - \tfrac{1}{2})\},$$

and

$$\left(\frac{\partial C_H}{\partial \eta}\right)_{\alpha=\eta=0} = -\frac{1}{2\pi\beta_\infty E^2}\left(\frac{4a}{Uc}\right)^2\sin\lambda_m \times \{(\pi - \lambda_m)(1 - \cos\lambda_m) - \sin\lambda_m(1 + \cos\lambda_m - 2\cos\lambda'_m)\}.$$

Now/

---

*This expression neglects a very small term due to the dependence of the limits of the integrals in equation (55) on η.

Now

$$C_H = (C_H)_{\alpha=\eta=0} + \alpha\left(\frac{\partial C_H}{\partial \alpha}\right)_{\alpha=\eta=0} + \eta\left(\frac{\partial C_H}{\partial \eta}\right)_{\alpha=\eta=0},$$

and using equation (40) we have

$$C_H = \left\{(C_H)_{\alpha=\eta=0} + \alpha_o\left(\frac{\partial C_H}{\partial \alpha}\right)_{\alpha=\eta=0}\right\} + \alpha'\left(\frac{\partial C_H}{\partial \alpha}\right)_{\alpha=\eta=0}$$
$$+\eta\left\{\left(1 - \frac{\lambda_m}{\pi} + \frac{\sin\lambda_m}{\pi}\right)\left(\frac{\partial C_H}{\partial \alpha}\right)_{\alpha=\eta=0} + \left(\frac{\partial C_H}{\partial \eta}\right)_{\alpha=\eta=0}\right\}.$$

Comparing this equation with (12), and using the values of the derivatives found above, we conclude that

$$\left.\begin{aligned} b_o &= (C_H)_{\alpha=\eta=} - \frac{\alpha_o}{\beta_\infty E^2}\left(\frac{4a}{Uc}\right)^2 \\ &\qquad \{\sin\lambda_m(1 + \tfrac{1}{2}\cos\lambda_m - \cos\lambda_m') + (\pi - \lambda_m)(\cos\lambda_m' - \tfrac{1}{2})\} \\ b_1 &= -\frac{1}{\beta_\infty E^2}\left(\frac{4a}{Uc}\right)^2 \{\sin\lambda_m(1 + \tfrac{1}{2}\cos\lambda_m - \cos\lambda_m') + (\pi - \lambda_m)(\cos\lambda_m' - \tfrac{1}{2})\} \\ b_2 &= -\frac{1}{\pi\beta_\infty E^2}\left(\frac{4a}{Uc}\right)^2 \{(\pi - \lambda_m)\sin\lambda_m + \tfrac{1}{2}\sin^2\lambda_m - (\tfrac{1}{2} - \cos\lambda_m')(\pi - \lambda_m)^2\}, \end{aligned}\right\} \quad \ldots(58)$$

while from $C_L = a_1\alpha$ and the definition of b we have

$$b = \frac{1}{2\pi\beta_\infty E^2}\left(\frac{4a}{Uc}\right)^2 \sin\lambda_m\{(\pi - \lambda_m)(1 - \cos\lambda_m) - \sin\lambda_m(1 + \cos\lambda_m - 2\cos\lambda_m')\}. \quad \ldots(59)$$

4. Summary of Formulae

The formulae given in Section 3 for the control characteristics are of two types:- (a) the accurate integral formulae, such as equations (57), and (b) the approximations, such as equations (58). The integral formulae are relatively simple to apply, particularly if q/U is calculated by the polygon method, but they do involve a few hours computation. The author considers that they are sufficiently accurate for most purposes for aerofoils of thickness ratio less than 20% travelling at speeds such that $M_\infty < M_{crit.}$ The approximations, which

will/

will be summarized below, will, in the author's opinion, give reliable results for aerofoils of thickness ratio less than, say 10%, when $M_\infty < (M_{crit.} - 0.2)$. As far as thickness effects are concerned it appears from the example in the next section that these approximations are more accurate than the method given in Ref. 3 called "Approximation III, Simple Theory", which involves numerical integration as in the author's more accurate method.

The rate of change of the no-lift angle is given by

$$\left(\frac{\partial(\alpha_0' - \alpha_0)}{\partial\eta}\right)_{\eta=0} = 1 - \frac{\lambda_n}{\pi} + \frac{\sin\lambda_n}{\pi}, \qquad \ldots(36)$$

where, from equation (27) λ_n satisfies

$$\cos\lambda_n = 1 - 2\left(\frac{cU}{4a}\right)\int_0^{\bar{s}/c} \frac{q}{U}\, d\left(\frac{s}{c}\right), \qquad \ldots(60)$$

in which $\bar{s}$ is the distance from the front stagnation point to the commencement of the flap. The ratio $(4a/Uc)$ is given approximately by (equation (90), Appendix II)

$$\left(\frac{4a}{Uc}\right) = 1 + \frac{1}{2\pi\beta_\infty}\int_0^c \frac{y_u - y_\ell}{x(c - x)}\, dx, \qquad \ldots(61)$$

(the suffices u and ℓ referring to the upper and lower surfaces respectively) or alternatively, from equation (27)

$$\left(\frac{4a}{Uc}\right) = \int_0^{p/c} \frac{q}{U}\, d\left(\frac{s}{c}\right). \qquad \ldots(62)$$

In equations (60) and (62) s/c can be replaced by x/c for thin aerofoils.

The numbers a_0, a_1 and a_2 are given by

$$a_0 = \frac{2\pi}{\beta_\infty}\left(\frac{4a}{Uc}\right)\alpha_0, \qquad \ldots(50)$$

$$a_1 = \frac{2\pi}{\beta_\infty}\left(\frac{4a}{Uc}\right), \qquad \ldots(51)$$

and $$a_2 = a_1\left(1 - \frac{\lambda_n}{\pi} + \frac{\sin\lambda_n}{\pi}\right), \qquad \ldots(52)$$

where/

where α_0 is given approximately by (equation (91), Appendix II)

$$\alpha_0 = \left(\frac{Uc}{4a}\right)^{3/2} \frac{1}{\pi} \int_0^c \frac{(y_u + y_\ell)}{x^{\frac{1}{2}}(c - x)^{3/2}} \, dx \, . \qquad \ldots(63)$$

The derivatives h and m_0 (equations (54)) are given by

$$h = \frac{1}{4}\left(\frac{4a}{Uc}\right), \qquad \ldots(64)$$

and $$m_0 = \frac{1}{2\beta_\infty}\left(\frac{4a}{Uc}\right)^2 \sin\lambda_m(1 - \cos\lambda_m) \, , \qquad \ldots(65)$$

while b_0, b, b_1 and b_2 are given by equations (58) and (59) of the previous section. Usually it is sufficient to write $\lambda_m' = \lambda_m$, when the equations for b_1 and b become

$$b_1 = -\frac{1}{E^3\beta_\infty}\left(\frac{4a}{Uc}\right)^2 \left\{\sin\lambda_m(1 - \tfrac{1}{2}\cos\lambda_m) - (\pi - \lambda_m)(\tfrac{1}{2} - \cos\lambda_m)\right\}, \ldots(66)$$

and $$b = \frac{1}{2E^2\beta_\infty}\left(\frac{4a}{Uc}\right)^2 \sin\lambda_m\left(1 - \frac{\lambda_m}{\pi} - \frac{\sin\lambda_m}{\pi}\right)(1 - \cos\lambda_m) \, . \qquad \ldots(67)$$

The derivative b_2 then follows from equation (13).

The equations given above for the control derivatives differ from those given by Glauert[2] only by

(i) the compressibility term, $1/\beta_\infty$,

(ii) the 'thickness' term, $\left(\frac{4a}{Uc}\right)$, and

(iii) the meaning to be assigned to λ_m. (λ_m is the angular co-ordinate of the hinge in the (ϕ,ψ) plane in the author's theory, whereas in Glauert's theory λ_m is the angular co-ordinate of the hinge in the (x,y) plane.)

In Ref. 11 Perring extended Glauert's flat plate theory to plates with multiply-hinged flaps. The analysis of this paper is easily extended to aerofoils with such flaps. If Perring's results are modified as described in (i), (ii) and (iii) above there will result the author's approximate equations for this type of flap.

5. Examples

(a) An Example given in Ref. 3

Goldstein and Preston gave as an example of their method, the calculation of b, b_1 and b_2 for a symmetrical "roof-top" aerofoil for which the velocity distribution is defined to be

$$\frac{q}{U} = \begin{cases} 1.1337 + 0.1213\,x & 0 \leqslant x \leqslant 0.6 \\ 1.2064 - 0.9706(x - 0.6) & 0.6 \leqslant x \leqslant 1.0\,. \end{cases}$$

The flap commences at $x/c = 0.8$, and the flow is incompressible. If it is assumed that $x/c \doteq s/c$ in equations (60) and (62), then from the given velocity distribution (normally this would have to be calculated as a first step), it is easily found that

$$\lambda_m = 132°1', \quad \text{and} \quad \frac{4a}{Uc} = 1.1070\,.$$

$$\left(\text{In Glauert's theory } \lambda_m = 180° - \cos^{-1}(0.6) = 126°52', \text{ and } \frac{4a}{Uc} = 1.\right)$$

Thus from equations (63), (36), (50), (51), (52), (64), (65), (66), (67) and (13) we find respectively

$$\alpha_0 = 0, \left(\frac{\partial\alpha_0'}{\partial\eta}\right)_{\eta=0} = 0.503, a_0 = 0, a_1 = 6.956, a_2/a_1 = 0.503,$$

$$h = 0.277, m_0 = 0.760, b_1 = -0.376, b = 0.572, \text{ and } b_2 = -0.763\,.$$

The values of b_1, b_2 and b given in Ref. 3 are compared with those given above in the following table.

$-b_1$	$-b_2$	b	Method
0.450	0.923	0.648	Ref. 3 { Glauert
0.349	0.739	0.547	Ref. 3 { Approx. III (simple theory)
0.364	0.774	0.574	Ref. 3 { Approx. III (complex theory)
0.376	0.763	0.572	Theory of this paper

The approximate theory of this paper appears from this example to be very satisfactory, particularly as this aerofoil is 15% thick.

(b) Aerofoil RAE 104 at $M_0 = 0.7$

The compressible about the symmetrical aerofoil, RAE 104 was calculated in Ref. 5 by the polygon method. The following figures taken from Table 6 of that report apply to $M_0 = 0.7$.

Table/

y°	0	3	9	15	21	27	35	45	55	65
x/c	0	0.0008	0.0067	0.0177	0.0337	0.054	0.087	0.140	0.204	0.275
q/U	0	0.470	0.883	1.051	1.110	1.139	1.156	1.176	1.178	1.180

y°	75	85	95	105	115	125	135	145	155	165
x/c	0.352	0.433	0.516	0.597	0.676	0.755	0.827	0.889	0.941	0.977
q/U	1.181	1.179	1.178	1.167	1.110	1.053	1.005	0.967	0.927	0.878

y°	175	180
x/c	0.997	1.000
q/U	0.790	0

$$\left(\frac{4a}{Uc}\right) = 1.1200$$

We shall calculate the control characteristics for a flap commencing at $x/c = 0.75$. By interpolation in the above figures we find that at $x/c = 0.75$, $\gamma = \lambda_n = 125°40'$. Also $1/\beta_{\infty} = 1.4003$, and hence from the equations given in Section 4 we find that

$$a_o = 0, \left(\frac{\partial a_o'}{\partial \eta}\right)_{\eta=0} = 0.561, \; a_o = 0, \; a_1 = 9.854, \; a_2/a_1 = 0.561,$$

$$h = 0.280, \; m_o = 1.129, \; b_1 = -0.624, \; b = 0.783, \; b_2 = -1.133.$$

References

No.	Author(s)	Title, etc.
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APPENDIX I/

APPENDIX I

An Exact Method of Calculating a_1 and h in Incompressible Flow

The origin in the z plane will be taken at the 'centre of the aerofoil profile'[8], which is defined by the equation

$$\lim_{\omega \to \infty} (Uz - w) = 0. \qquad \ldots(68)$$

It is shown in Ref. 4, §19 that

$$Uz = w - \frac{iU}{2\pi} \int_{-\pi}^{\pi} y(\gamma^*) \coth \tfrac{1}{2}(\zeta - i\gamma^*) d\gamma^*, \qquad \ldots(69)$$

the conjugate equation to which is

$$Uz = w - \frac{U}{2\pi} \int_{-\pi}^{\pi} \left\{ x(\gamma^*) - \frac{\phi}{U} \right\} \coth \tfrac{1}{2}(\zeta - i\gamma^*) d\gamma^*,$$

where $x(\gamma^*)$, $y(\gamma^*)$ are the aerofoil co-ordinates. By addition of these results, and taking $\lim_{-\zeta \to \infty}$, which is equivalent to $\lim_{\omega \to \infty}$ (equation (6)), we find that the origin must be taken in the z plane so that

$$\int_{-\pi}^{\pi} x(\gamma^*) d\gamma^* = \int_{-\pi}^{\pi} y(\gamma^*) d\gamma^* = 0. \qquad \ldots(70)$$

If the axis $x = 0$ is taken to satisfy equation (21), then the z plane is completely fixed in position.

If (X,Y) is the force acting on the aerofoil, and M is the nose-up moment about the origin (defined by (70)), then the theorem of Blasius[8] is that

$$X - iY = \tfrac{1}{2}i\rho \int_c \left(\frac{dw_\alpha}{dz_\alpha}\right)^2 dz_\alpha, \qquad M + iN = \tfrac{1}{2}\rho \int_c z_\alpha \left(\frac{dw_\alpha}{dz_\alpha}\right)^2 dz_\alpha,$$

i.e., from equations (5) and (16)

$$X - iY = \tfrac{1}{2}i\rho e^{-i\alpha}U \int_c e^{-2f_\alpha + f} dw, \quad M + iN = \tfrac{1}{2}\rho e^{-2i\alpha}U \int_c z e^{-2f_\alpha + f} dw, \qquad \ldots(71)$$

where/

where C is any closed contour about the aerofoil. The only contributions to these integrals arise from the coefficients of $1/w$ in the expansions of the integrands. Consider first the force (X,Y). From equations (23) and (71) we find

$$X - iY = -\pi a\rho U\left\{4i \sin\alpha - \frac{e^{-i\alpha}}{\pi}\int_{y^*=-\pi}^{\pi} e^{-iy^*}\, d\theta(y^*)\right\},$$

and since this must vanish when $\alpha = 0$, we have an alternative proof of equations (24). Thus

$$X = 0,\ Y = 4a\pi\rho U \sin\alpha,$$

and the lift coefficient is given by

$$C_L = \frac{Y}{\frac{1}{2}\rho c\, U^2} = 2\pi\left(\frac{4a}{Uc}\right)\sin\alpha . \qquad \ldots(72)$$

This equation is a well-known text book result, but the corresponding result for C_m given below is possibly new.

From equation (71)

$$M + iN = +\rho\pi i e^{-2i\alpha} \times \left(\text{coef. of } \frac{1}{w} \text{ in } Uze^{-2f_\alpha+f}\right). \qquad \ldots(73)$$

Equations (16), (23) and (24) yield

$$f = \log\left(\frac{Udz}{dw}\right) = \frac{a^2}{2\pi w^2}\int_{y^*=-\pi}^{\pi} e^{-2iy^*}\, d\theta(y^*) + O\left(\frac{a}{w}\right)^3,$$

and hence with the aid of equation (68), we have

$$Uz = w - \frac{a^2}{2\pi w}\int_{y^*=-\pi}^{\pi} e^{-2iy^*}\, d\theta(y^*) + O\left(\frac{a}{w}\right)^2 . \qquad \ldots(74)$$

From equations (23) and (24) it follows that

$$e^{-2f_\alpha+f} = 1 + \frac{4ai}{w}e^{+i\alpha}\sin\alpha - \frac{a^2}{w^2}$$

$$\times\left\{2ie^{+2i\alpha}\sin 2\alpha + 8e^{+2i\alpha}\sin^2\alpha + \frac{1}{2\pi}\int_{y^*=-\pi}^{\pi} e^{-2iy^*}\, d\theta(y^*)\right\} + O\left(\frac{a}{w}\right)^3 . \qquad \ldots(75)$$

Now/

Now $C'_m = \dfrac{M}{\frac{1}{2}\rho c^2 U^2}$, where C'_m is the moment coefficient about the origin defined by equation (70), and so from (73), (74) and (75) it follows that

$$C'_m = \frac{\pi}{4}\left(\frac{4a}{Uc}\right)^2 \sin 2\alpha$$

$$\times \left\{1 - \frac{1}{2\pi}\int_{y^*=-\pi}^{\pi} \cos 2y^* \, d\theta(y^*) - \frac{\cot 2\alpha}{2\pi}\int_{y^*=-\pi}^{\pi} \sin 2y^* \, d\theta(y^*)\right\}. \qquad \ldots(76)$$

The conjugate equation to this was given by Lighthill[9] for application to the problem of aerofoil design.

An alternative form of this equation can be found thus. From equations (6) and (69)

$$U\frac{dz}{dw} = 1 - \frac{Ui}{8a\pi \sinh \zeta}\int_{-\pi}^{\pi} y(y^*) \operatorname{cosech}^2 \tfrac{1}{2}(\zeta - iy^*) dy^*$$

$$= 1 + \frac{iaU}{\pi w^2}\int_{-\pi}^{\pi} e^{iy^*} y(y^*) dy^* + 0\left(\frac{a}{w}\right)^3,$$

$$\text{i.e., } \log\left(\frac{Udz}{dw}\right) = f = \frac{-iaU}{\pi w^2}\int_{-\pi}^{\pi} y(y^*) \, e^{iy^*} dy^* + 0\left(\frac{a}{w}\right)^3.$$

Comparing this equation with (23) (with $\alpha = 0$) we conclude that

$$\int_{y^*=-\pi}^{\pi} \cos 2y^* \, d\theta(y^*) = 8\left(\frac{Uc}{4a}\right)\int_{-\pi}^{\pi} \frac{y}{c}(y^*) \sin y^* \, dy^* \qquad \ldots(77)$$

and

$$\int_{y^*=-\pi}^{\pi} \sin 2y^* \, d\theta(y^*) = -8\left(\frac{Uc}{4a}\right)\int_{-\pi}^{\pi} \frac{y}{c}(y^*) \cos y^* \, dy^*. \qquad \ldots(78)$$

Thus equation (76) can be written in the form

$$C'_m = \frac{\pi}{4}\left(\frac{4a}{Uc}\right)^2 \sin 2\alpha$$

$$\times\left\{1 - \frac{4}{\pi}\left(\frac{Uc}{4a}\right)\int_{-\pi}^{\pi} \frac{y}{c}(y^*) \sin y^* \, dy^* + \frac{4 \cot 2\alpha}{\pi}\left(\frac{Uc}{4a}\right)\int_{-\pi}^{\pi} \frac{y}{c}(y^*) \cos y^* \, dy^*\right\}.$$

$$\ldots(79)$$

If/

If the polygon method of finding the velocity distribution about the aerofoil has been used then the functions $\frac{x}{c}(y^*)$, $\frac{y}{c}(y^*)$, $\theta(y^*)$ and $(4a/Uc)$ will be immediately available, and C_m and C_L can be calculated directly.

Suppose the centre of the profile lies at a distance $\bar{x}$ then

$$h = \bar{x} - \left(\frac{\partial C_m'}{\partial C_L}\right)_{\alpha=0} \text{, approximately}$$

i.e., from (79),

$$h = \bar{x} - \frac{1}{4}\left(\frac{4a}{Uc}\right)\left\{1 - \frac{4}{\pi}\left(\frac{Uc}{4a}\right)\int_{-\pi}^{\pi}\frac{y}{c}(y^*)\sin y^*\,dy^*\right\}. \qquad \ldots(80)$$

If we write $\quad Ux \doteqdot 2a + \phi = 2a(1 - \cos y)$, $\qquad \ldots(81)$

$$h \doteqdot \bar{x} - \frac{1}{4}\left(\frac{4a}{Uc}\right)\left\{1 - \frac{8A}{\pi c^2}\left(\frac{Uc}{4a}\right)^2\right\}, \qquad \ldots(82)$$

where A is the area of the aerofoil, but this equation requires knowing $\bar{x} - \left(\frac{4a}{Uc}\right)$. The numbers $\left(\frac{4a}{Uc}\right)$ and $\bar{x}$ are discussed in Appendices II and III respectively.

APPENDIX II

The Value of $\left(\frac{4a}{Uc}\right)$

This important ratio occurs throughout the theory. In the polygon method[5] it is calculated as an essential step from (c.f. equation (26))

$$\frac{p}{c}\left(\frac{cU}{4a}\right) = \frac{1}{2}\int_0^{\pi}\frac{U\sin y}{q}\,dy, \qquad \ldots(84)$$

where/

where p is the distance between the stagnation points measured along the upper surface. Integration of equation (18) by parts results in

$$r(\gamma) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta(\gamma^*) \cot \tfrac{1}{2}(\gamma^* - \gamma)\, d\gamma^* ,$$

on the aerofoil surface, and the approximation (25) then yields

$$\frac{U}{q} \doteqdot e^{r/\beta_\infty} \doteqdot 1 + \frac{1}{2\pi\beta_\infty} \int_{-\pi}^{\pi} \theta(\gamma^*) \cot \tfrac{1}{2}(\gamma^* - \gamma)\, d\gamma^* . \qquad \ldots(85)$$

For aerofoils of moderate thickness $p \doteqdot c$, and hence from (84) and (85)

$$\left(\frac{cU}{4a}\right) = 1 + \frac{1}{2\pi\beta_\infty} \int_{-\pi}^{\pi} \theta(\gamma^*) \sin \gamma^* \log \left| \tan \tfrac{1}{2}\gamma^* \right| d\gamma^* . \qquad \ldots(86)$$

If we make use of the approximation (81) then

$$\int \theta(\gamma^*) \sin \gamma^*\, d\gamma^* \doteqdot \left(\frac{U}{2a}\right) \int \frac{dy}{dx}\, dx = \left(\frac{Uc}{4a}\right) \frac{2y}{c} ,$$

and so integrating (86) by parts we have

$$\left(\frac{4a}{Uc}\right) \doteqdot 1 + \frac{1}{\pi\beta_\infty} \int_{-\pi}^{\pi} \left(\frac{y}{c}\right) \frac{d\gamma^*}{\sin \gamma^*} . \qquad \ldots(87)$$

It can be shown from equation (69) that this equation is exact in incompressible flow.

From (87) it follows that the effect of compressibility on $\left(\frac{4a}{Uc}\right)$ is given by

$$\left(\frac{4a}{Uc}\right) = 1 + \frac{1}{\beta_\infty} \left\{ \left(\frac{4a}{Uc}\right)_i - 1 \right\} , \qquad \ldots(88)$$

where $\left(\frac{4a}{Uc}\right)_i$ is the value in incompressible flow. Thus, for example, a_1 (equation (51)) is related to $(a_1)_i$ by

$$a_1 = \frac{2\pi}{\beta_\infty} \left\{ 1 + \frac{1}{\beta_\infty} \left(\frac{(a_1)_i}{2\pi} - 1 \right) \right\} . \qquad \ldots(89)$$

A/

A useful approximation for $\left(\frac{4a}{Uc}\right)$ follows from (81) and (87). If y_u and y_ℓ denote values of y at opposite points on the upper and lower surface respectively, then we find

$$\left(\frac{4a}{Uc}\right) \doteq 1 + \frac{1}{2\pi\beta_\infty} \int_0^c \frac{y_u - y_\ell}{x(c - x)} dx . \qquad ...(90)$$

Approximations to many of the equations given in this paper can be found by using equation (81). For example consider equation (22) for α_0. Making use of equations (24), which are clearly independent of the origin of $\underset{\sim}{\theta}$, we can write

$$\alpha_0 = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \underset{\sim}{\theta}(\gamma^*) \{1 - \cos \gamma^*\} d\gamma^* ,$$

which after some calculation reduces to the approximate form

$$\alpha_0 = \left(\frac{Uc}{4a}\right)^{3/2} \frac{1}{\pi} \int_0^c \frac{y_u + y_\ell}{x^{\frac{1}{2}}(c - x)^{3/2}} dx . \qquad ...(91)$$

When $(Uc/4a)$ is taken equal to unity this equation is in agreement with the usual formula of thin aerofoil theory[10].

APPENDIX III

An Approximation for h

If the centre of the profile is at a distance $\bar{x}$ from the leading edge, then taking the origin of the (x,y) plane at the leading edge, we find from equation (70) that

$$\bar{x} = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(\gamma) \, d\gamma .$$

An/

An integration by parts results in

$$\bar{x} = c - \frac{1}{2\pi}\int_{-\pi}^{\pi} y\,\frac{dx}{ds}\,\frac{ds}{d\phi}\,\frac{d\phi}{dy}\,dy$$

$$\doteqdot c - \frac{a}{\pi U}\int_{-\pi}^{\pi}\frac{U}{q}\,y\sin y\,dy\,,$$

since $\quad \dfrac{d\phi}{dy} = 2a\sin y,\ \dfrac{dx}{ds} \doteqdot 1 \quad \text{and} \quad \dfrac{ds}{d\phi} = \dfrac{1}{q}.$

If the value of U/q from the incompressible form of equation (85) is now substituted in this equation for $\bar{x}$, then with the aid of (24), it is found that

$$\bar{x} = c - \frac{2a}{U} + \frac{2a}{\pi U}\int_{-\pi}^{\pi}\theta(y)\sin y\log\cos\tfrac{1}{2}y\;dy\,.$$

Writing $\theta \doteqdot dy/dx$, and integrating by parts we have

$$\bar{x}/c = 1 - \tfrac{1}{2}\left(\frac{4a}{Uc}\right) + \frac{1}{2\pi}\int_{-\pi}^{\pi} y(y)\tan\tfrac{1}{2}y\;dy\,. \qquad \ldots(92)$$

Finally from equations (80), (87) and (92) it follows that

$$h_1 = \frac{1}{4} + \frac{1}{2\pi}\int_{0}^{\pi}\left(\frac{y_u}{c} - \frac{y_\ell}{c}\right)\left(2\sin y - \frac{1 + 2\cos y}{2\sin y}\right)dy\,. \qquad \ldots(93)$$

In evaluating the integral it is usually sufficient to write

$$\cos y = \left(1 - \frac{2x}{c}\right).$$

When h_1, the incompressible flow value of h, has been found from (93), it follows from (c.f. equation (89))

$$h = \frac{1}{4} + \frac{1}{\beta_\infty}\left(h_1 - \frac{1}{4}\right). \qquad \ldots(94)$$

APPENDIX IV

The Exact Theory of the Hinged Flat Plate in Incompressible Flow

Fig. 4(a) shows the flat plate at the no-lift position, while Fig. 4(b) shows the relation (θ, γ), which should not be confused with the relation shown in Fig. 3(b) where the meaning of γ is slightly different.

Equations (24) lead to $\quad \sin \tfrac{1}{2}\lambda = \dfrac{\eta}{\pi} \sin \lambda_m$

and $\quad \lambda = \lambda_1 - \lambda_0,$

where $\quad \lambda_m = \tfrac{1}{2}(\lambda_1 + \lambda_0),$

while equation (21) leads to the value

$$\alpha_0' = \eta \left(1 - \frac{\lambda_m}{\pi} + \frac{\sin \lambda_m}{\pi}\right)$$

for the no-lift angle. From equations (16) and (18) we find that the velocity distribution is given by

$$\frac{q}{U} = \frac{\sin \tfrac{1}{2}\gamma}{\sin \tfrac{1}{2}(\gamma + \lambda)} \left[\frac{\sin \tfrac{1}{2}(\gamma + \lambda_1)}{\sin \tfrac{1}{2}(\gamma - \lambda_0)}\right]^{\eta/\pi},$$

and hence from equation (26) the (s, γ) relation is given by

$$\frac{sU}{4a} = \int_0^{\gamma} \cos \tfrac{1}{2}\gamma \sin \tfrac{1}{2}(\gamma + \lambda) \left[\frac{\sin \tfrac{1}{2}(\gamma - \lambda_0)}{\sin \tfrac{1}{2}(\gamma + \lambda_1)}\right]^{\eta/\pi} d\gamma .$$

Substitution of the (θ, γ) relation in equation (76) leads to

$$C_n' = \frac{\pi}{4}\left(\frac{4a}{Uc}\right)^2 \left\{\sin 2\alpha - \cos(2\alpha - \lambda) \left(\sin \lambda - \frac{\eta}{\pi} \sin 2\lambda_m\right)\right\},$$

and so

$$\left(\frac{\partial C_n'}{\partial \alpha}\right)_{\eta=\alpha=0} = \frac{\pi}{2}$$

$$\left(\frac{\partial C_n'}{\partial \eta}\right)_{\eta=\alpha=0} = -\tfrac{1}{2} \sin \lambda_m(1 - \cos \lambda_m),$$

as it is easily shown from equation (86) that $cU/4a = 1 + O(\eta^2)$.

APPENDIX V

Basic Mathematical Theory

The theory is based on the equations[7]

$$\frac{\partial\theta}{\partial n} + (1 - M^2)\frac{1}{q}\frac{\partial q}{\partial s} = 0, \quad \frac{\partial\theta}{\partial s} - \frac{1}{q}\frac{\partial q}{\partial n} = 0,$$

which with the aid of equations (1), (2) and the transformation

$$dr = (1 - M^2)^{\frac{1}{2}} d\left(\log \frac{U}{q}\right),$$

can be written in the form

$$\frac{\partial\theta}{\partial\psi} - m\frac{\partial r}{\partial\phi} = 0, \quad \frac{\partial\theta}{\partial\phi} + \frac{1}{m}\frac{\partial r}{\partial\psi} = 0. \qquad \ldots(95)$$

From (2) it is readily found that in subsonic flow

$$m = m_\infty\left\{1 + \frac{\nu+1}{2\beta_\infty^2} M_\infty^4 \left(\frac{q}{U} - 1\right) + 0\left[\left(\frac{q}{U} - 1\right)^2\right]\right\},$$

so that for thin aerofoils $\left(\frac{q}{U} \approx 1\right)$ at high subsonic Mach numbers or thick aerofoils at lower subsonic Mach numbers, von Kármán's approximation

$$m = m_\infty, \qquad \ldots(96)$$

is plausible. This approximation enables (95) to be written as the Cauchy-Riemann equations

$$\frac{\partial\theta}{\partial(m_\infty\psi)} - \frac{\partial r}{\partial\phi} = 0, \quad \frac{\partial\theta}{\partial\phi} + \frac{\partial r}{\partial(m_\infty\psi)} = 0.$$

Since, in any application we shall make, these four derivatives exist and are continuous in the open domain outside the aerofoil contour, we can write

$$r + i\theta \equiv f_\alpha(\phi + im_\infty\psi),$$

or if $w_\alpha = \phi + im_\infty\psi$,

$$f_\alpha = f_\alpha(w_\alpha), \qquad \ldots(97)$$

where the suffix α denotes the appropriate incidence (measured from the no-lift angle).

A particular case of (97) is the no-lift solution

$$f = f(w). \qquad \ldots(98)$$

/Now

Now for small angles if incidence (only such angles are important in the paper), we make the assumption that w_α is an analytic function of w, i.e., that (97) can be written

$$f_\alpha = f_\alpha(w). \qquad \ldots(99)$$

for incompressible flow (99) is exactly true, since both w and w_α are analytic functions of z. It is important to notice that the approximation involved in (99) is merely one of the location of the solution f_α, and it is similar in character to the approximation commonly made in engineering applications of the Kármán-Tsien method (cf. reference 7, p.183). The approximation receives some experimental verification in reference 5. Further verification of its plausibility is to be found in the approximate equations of section 4, where it yields the same compressibility factor, $1/\beta_\infty$, as that predicted by the linear perturbation theory.

It can be verified that the modified definition of r given by equation (3) is consistent with the approximation (96). It is an empirical modification made, because as shown in reference 5, it leads to improved agreement with experiment.

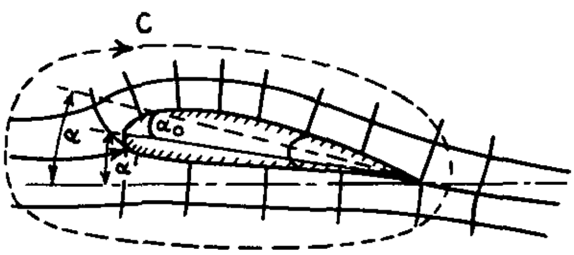
With the aid of equation (6) it is found that the value of f given by equation (18) satisfies equation (97) and the appropriate boundary conditions. When the aerofoil is placed at an angle of incidence α, on the aerofoil surface θ_α is given by

$$\theta_\alpha(\ *) = \begin{cases} \theta(y^*) - \alpha, & -\pi \leqslant y^* \leqslant \pi \\ +\pi \quad , & -y_0 \leqslant y^* \leqslant 0, \end{cases} \qquad \ldots(100)$$

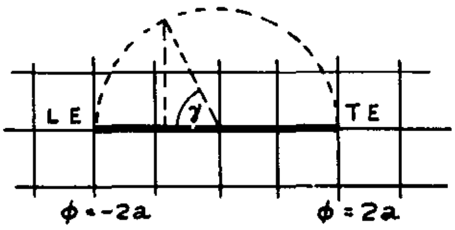
where the π term is due to the reversal in flow direction caused by the displacement of the front stagnation point from $y^* = 0$ to $y^* = -y_0$. (By the Joukowski Hypothesis the position of the rear stagnation point is unchanged.) The value of y_0 is fixed by the condition that the flow at infinity must be undisturbed. It is not difficult to verify that f_α given by (19) satisfies equation (99), the boundary conditions (100) and leaves the flow at infinity undisturbed. Full details of the proof of these results from equation (99), is to be found in reference 4.

FIGS 1-4

FIG 1

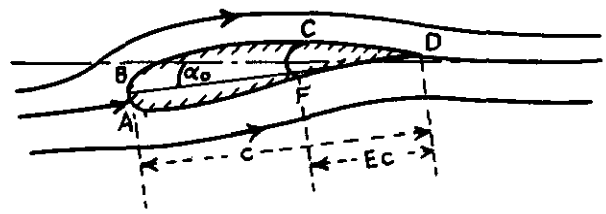


(a) z_α plane

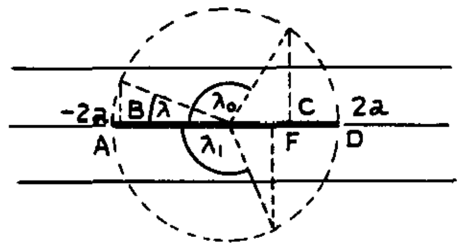


(b) ω plane (zero circulation)

FIG 2

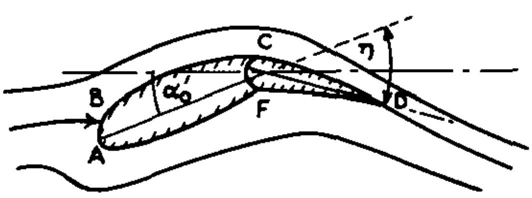


(a) z plane, $\eta = 0$

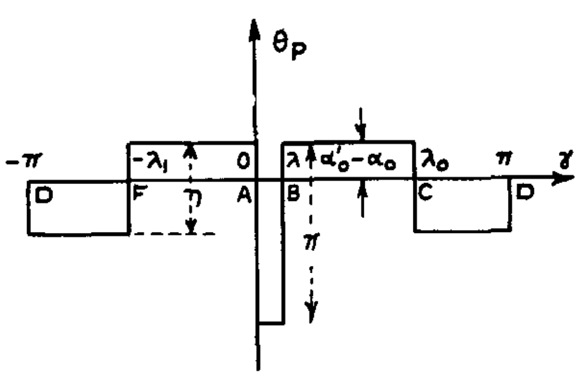


(b) ω plane

FIG 3

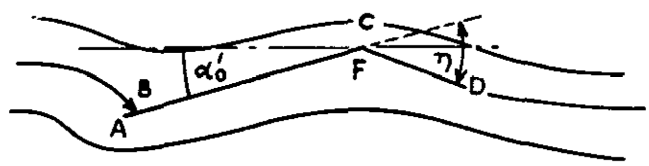


(a) z plane, $\eta \neq 0$

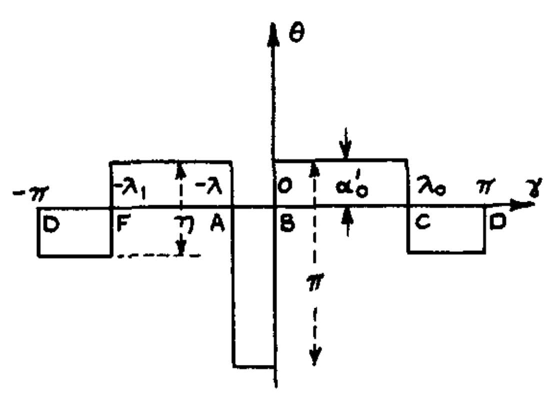


(b) (θ_p, γ)

FIG. 4



(a) z plane



(b) (θ, γ)

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