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The Buckling of a Long Curved Panel under Axial Compression

By

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Summary.—In modern aeroplane design a detailed knowledge of the behaviour of thin panels is very necessary. The problem of flat panels may be regarded as largely solved. This is not true, however, of curved panels, and the object of this report is to obtain further information on the behaviour of curved panels under axial loading. The problem falls naturally into two parts, according as the investigation is concerned with the initial buckling of the panel or with its subsequent behaviour. The second part of the problem will be treated later, and attention is here confined to obtaining an accurate expression for the initial buckling stress of a slightly curved and perfectly formed panel, the two straight edges of which are either simply supported or fixed.

From the results obtained two main points stand out. Firstly, that for a perfectly formed panel the stabilising effect of curvature is very great, and for appreciable curvature it is almost immaterial whether the edges are simply supported or fixed. Secondly, that for appreciable curvature S. C. Redshaw's approximate solutions [1, 2] are sufficiently close to the exact solutions for all practical purposes.

The method of solution developed in this report is of very general application, and can be used to find the stability of slightly curved panels under combined shear and compression or tension. On the basis of this report and the work referred to in reference 3, it should be possible to investigate the post buckling behaviour of an axially loaded curved panel supported along its edges.

1. *Introduction*.—The part played by thin sheets of metal in modern aeroplane construction is now so important that a knowledge of their behaviour under various types of loading is very necessary. For flat panels the problem of determining the stress at which buckling occurs, and the stress distribution after buckling, may be regarded as solved. The same is not true, however, of curved panels, and the object of this report is to throw further light on the behaviour of curved panels under axial loading[4]. The problem falls naturally into two parts, according as the investigation is concerned with the initial buckling of the panel or with its subsequent behaviour. Attention is here confined to the matter of initial buckling.

In the course of the last few years many experiments have been carried out on curved panels under axial loading, and they have all emphasised the importance of initial irregularities [2, 4, 5, 6]. Although the effect of these in lowering the buckling load can only be estimated after considering the post buckling condition [3, 7], the stress at which a perfectly formed panel may be expected to become unstable is still important for two reasons. In the first place it is of considerable assistance in carrying out the post buckling investigation for which only approximate methods are at present available, and in the second place as indicating an ideal upper limit for perfectly formed perfectly loaded panels.

Besides the experimental work referred to above, the problem has been treated theoretically by S. C. Redshaw [1, 2] and S. Timoshenko[8]. As, however, their work is based on assuming forms for the displacements which satisfy either the boundary conditions or the equations of equilibrium but not both together, their solutions are only approximate, and the object of this report is to obtain an accurate expression for the buckling stress in the case of a slightly curved and perfectly

* R.A.E. Report, June, 1942.

formed strip whose straight edges are either simply supported or fixed. No effort has been made to satisfy the boundary conditions over the curved edges, but as the wave length of the buckles decreases rapidly with curvature, this limitation is not important unless the length of the panel is less than its width.

Attention is drawn to the method of solution developed in the Appendix for the case in which the edges are fixed, as it is extremely general and can be applied to find an accurate expression for the critical buckling stress of a slightly curved and perfectly formed strip whose edges are free, fixed or simply supported, and which is acted on by any combination of shear and compression or tension.

2. *Statement of Problem.*—The problem considered here is the stress at which a slightly curved and perfectly formed strip first starts to buckle under compression. The thickness and curvature of the strip are assumed constant. The loading is applied uniformly over the two curved edges, and the two straight edges are either simply supported or fixed.

3. *Description of Results.*—The method of solution is set out and explained in the Appendix, and the results are shown graphically in Figs. 1 and 2. Fig. 1 shows how the buckling stress varies with the ratio of bulge to thickness, and Fig. 2 gives the corresponding wave length of the buckles. In Fig. 1 the full lines indicate the exact solutions, and the broken lines show Redshaw's approximate solutions.

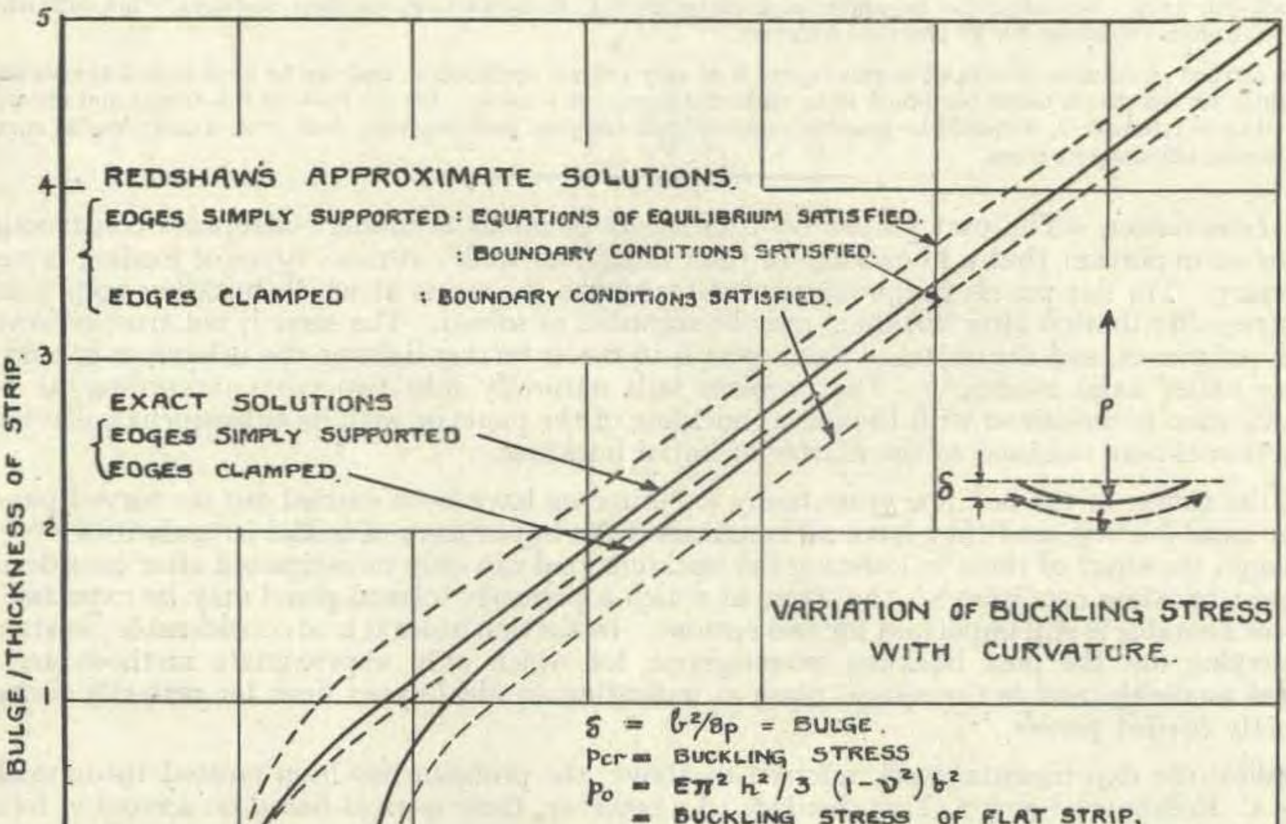

FIG. 1.

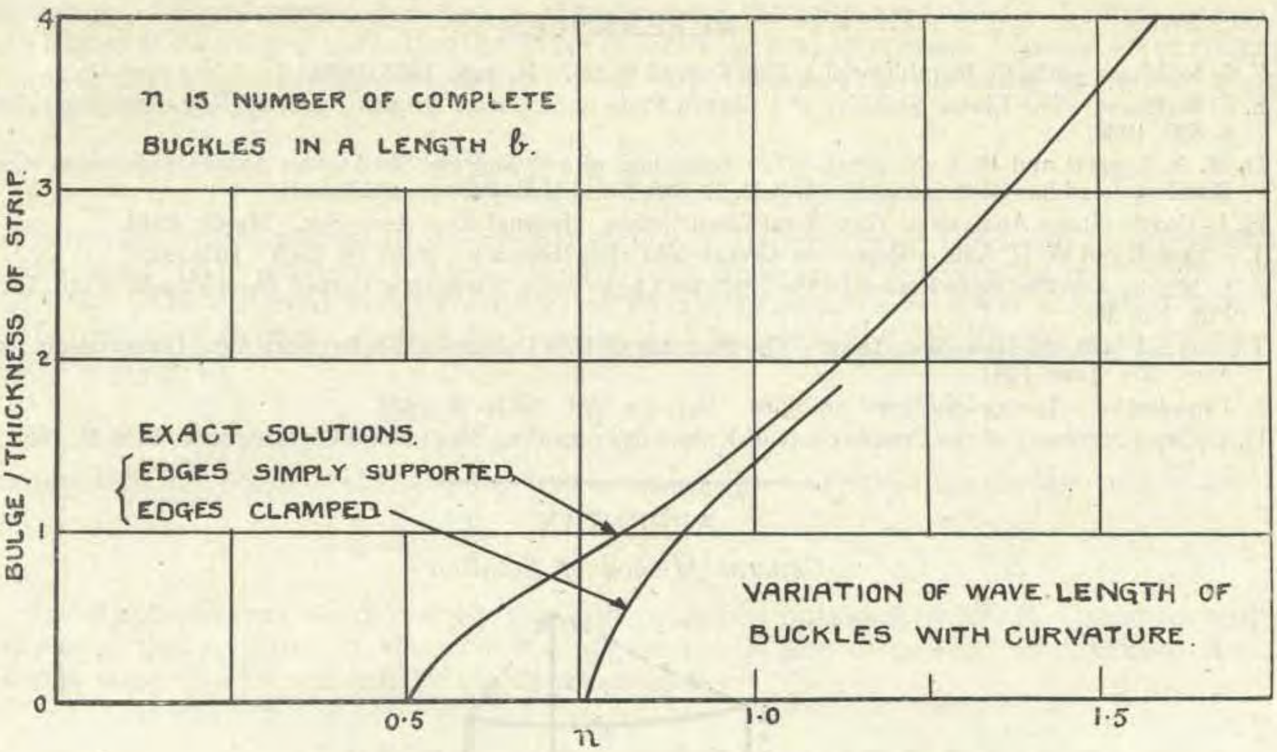

FIG. 2.

The results obtained in this report show that Redshaw's* approximate solutions are in general satisfactory. For the percentage error becomes appreciable only when the curvature is very small, and steadily decreases as the curvature is increased.

It is interesting to note that of Redshaw's two approximate solutions for the simply supported case, the one that satisfies the equations of equilibrium, but not the boundary conditions, gives too low a buckling stress, while the one that satisfies the boundary conditions, but not the equations of equilibrium, gives too high a value. Both these results are to be expected. For in the latter case failure to satisfy the equations of equilibrium presupposes the existence of constraints which are here such as to increase the buckling stress. Whereas in the former case the buckling stress is for edge conditions which are less restrictive than those in the actual problem considered.

4. *Conclusions and further development.*—From the above results two main points stand out. Firstly, that for a slightly curved and perfectly formed panel which is subjected to uniform axial loading the stabilising effect of curvature is very great, and for appreciable curvature it is almost immaterial whether the edges are simply supported or fixed. Secondly, that for appreciable curvature Redshaw's approximate solutions are sufficiently close to the exact solutions for all practical purposes.

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* In Redshaw's second paper[2] there is a slight error in the analysis for the particular case when the axial edges are fixed. It has entered through a slip in evaluating the strain energy function, and its effect is to replace the factor $(1 - \sigma)$ in his equation (53) by 1. As a result of this the value of K_{min} for Case (b) on p. 537 should now read

$$2\left\{\left(\frac{16}{3}+\frac{768d^2}{\pi^4 t^2}\right)^{\frac{1}{2}}+\frac{4}{3}\right\},$$

and it is this corrected expression which is plotted in Fig. 1. It should, however, be emphasised that the effect of this correction decreases with curvature, and for appreciable curvature is unimportant.

For a flat strip with fixed edges, Redshaw's corrected value for K_{min} is $7 \cdot 29$, which agrees with H. L. Cox's[9] result, while the value obtained by the precise method is $6 \cdot 94$.

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APPENDIX

General Method of Solution

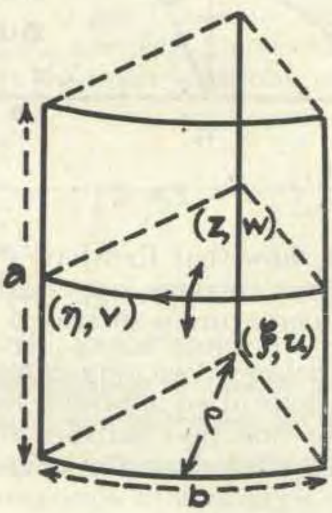

FIG. 3.

Notation.—

E = Young's modulus

ν = Poisson's ratio—assumed to be $0 \cdot 25$

$2h$ = thickness of strip

ρ = radius of curvature of strip

D = flexural rigidity, *i.e.* $2Eh^3/3(1 - \nu^2)$

a = length of strip

b = width of strip, measured along curved edge

p = axial compressive stress

p_0 = buckling stress of flat strip, edges simply supported

$\delta = \dfrac{b^2}{8\rho}$ = bulge

* This contains a very comprehensive list of references to theoretical and experimental work done on the buckling of curved panels.

Derivation of Fundamental Equations.—The edges of the strip are taken to be two generators and two lines of curvature, and attention is confined to the middle surface. Choosing the generator and line of curvature through the centre of the strip as axes of co-ordinates ξ, η; the edges of the strip are given by

$$\xi = \pm \frac{a}{2}, \eta = \pm \frac{b}{2},$$

and the inward drawn normal to the middle surface in its unstrained condition is introduced as a third axis z. With reference to these axes the displacements of any point in the middle surface are u, v, w. The external loading consists of an axial compressive stress p applied uniformly over the two curved edges, so that in Timoshenko's notation the equilibrium state of stress in the strip is given by

$$\sigma_\xi = -p, \qquad \sigma_\eta = 0, \qquad \tau_{\xi\eta} = 0,$$

Assuming that the edges of the strip are free to move, the corresponding displacements are

$$u = -\frac{p\xi}{E}, \qquad v = 0, \qquad w = -\frac{\nu p \rho}{E},$$

and these displacements clearly satisfy the shell equations obtained by W. R. Dean* for stability problems of this nature. If the above configuration is one of neutral equilibrium, the shell equations must also be satisfied by the displacements

$$u = -\frac{p\xi}{E} + u', \qquad v = v', \qquad w = -\frac{\nu p\rho}{E} + w',$$

where u', v', w' are arbitrarily small but not all zero. The three fundamental stability equations are now obtained by subtracting the shell equations for these two configurations and ignoring terms of order above the first in u', v', w'. The equations are very long, however, and as they can be easily simplified by making certain assumptions, it is not proposed to set them down in full. The assumptions referred to are that b/ρ is small, and that w' is of a larger order of magnitude than u' or v'. The former is implicit in the fact that the strip is only slightly bent, while the latter follows from considering the special case in which ρ is infinite. After carrying out the simplification just mentioned, the three equations reduce to

$$\frac{\partial}{\partial \xi}\left\{\frac{\partial u'}{\partial \xi} + \nu\left(\frac{\partial v'}{\partial \eta} - \frac{w'}{\rho}\right)\right\} + \frac{(1-\nu)}{2}\frac{\partial}{\partial \eta}\left(\frac{\partial v'}{\partial \xi} + \frac{\partial u'}{\partial \eta}\right) = 0, \quad \ldots \quad \ldots \quad (1)$$

$$\frac{(1-\nu)}{2}\frac{\partial}{\partial \xi}\left(\frac{\partial v'}{\partial \xi} + \frac{\partial u'}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left\{\left(\frac{\partial v'}{\partial \eta} - \frac{w'}{\rho}\right) + \nu\frac{\partial u'}{\partial \xi}\right\} = 0, \quad \ldots \quad \ldots \quad (2)$$

$$\frac{h^2}{3}\nabla^4 w' + (1-\nu^2)\frac{p}{E}\frac{\partial^2 w'}{\partial \xi^2} - \frac{1}{\rho}\left\{\left(\frac{\partial v'}{\partial \eta} - \frac{w'}{\rho}\right) + \nu\frac{\partial u'}{\partial \xi}\right\} = 0, \quad \ldots \quad \ldots \quad (3)$$

Introducing x and y defined by $x = \frac{\xi\pi}{b}$, $y = \frac{\eta\pi}{b}$, these equations take the form

$$\frac{\partial}{\partial x}\left[\frac{\partial u'}{\partial x} + \nu\left(\frac{\partial v'}{\partial y} - \frac{bw'}{\rho\pi}\right)\right] + \frac{(1-\nu)}{2}\frac{\partial}{\partial y}\left(\frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y}\right) = 0, \quad \ldots \quad \ldots \quad (4)$$

$$\frac{(1-\nu)}{2}\frac{\partial}{\partial x}\left(\frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y}\right) + \frac{\partial}{\partial y}\left[\left(\frac{\partial v'}{\partial y} - \frac{bw'}{\rho\pi}\right) + \nu\frac{\partial u'}{\partial x}\right] = 0, \quad \ldots \quad \ldots \quad (5)$$

$$\frac{h^2}{3}\nabla^4 w' + (1-\nu^2)\left(\frac{b}{\pi}\right)^2\frac{p}{E}\frac{\partial^2 w'}{\partial x^2} - \frac{b^3}{\pi^3\rho}\left[\left(\frac{\partial v'}{\partial y} - \frac{bw'}{\rho\pi}\right) + \nu\frac{\partial u'}{\partial x}\right] = 0, \quad \ldots \quad \ldots \quad (6)$$

* On the Theory of Elastic Stability. Proc. Roy. Soc. A. Vol. 107, page 734, 1925.

and it remains to find the relation between p, ρ, and the dimensions of the strip in order that equations (4) to (6) may have a non-zero solution satisfying the appropriate boundary conditions.

Introducing a stress function φ defined by

$$\frac{\partial u'}{\partial x} + \nu\left(\frac{\partial v'}{\partial y} - \frac{bw'}{\rho\pi}\right) = \pi\frac{\partial^2\varphi}{\partial y^2}, \quad \ldots \quad (7)$$

$$\frac{(1-\nu)}{2}\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) = \pi\frac{\partial^2\varphi}{\partial x\partial y}, \quad \ldots \quad (8)$$

$$\left(\frac{\partial v'}{\partial y} - \frac{bw'}{\rho\pi}\right) + \nu\frac{\partial u'}{\partial x} = \pi\frac{\partial^2\varphi}{\partial x^2}, \quad \ldots \quad (9)$$

equations (4) and (5) are satisfied identically, and then by eliminating u' and v' between (6), (7), (8) and (9), the equations for φ and w' are

$$\nabla^4\varphi + R\frac{\partial^2 w'}{\partial x^2} = 0\,, \quad \ldots \quad (10)$$

$$\nabla^4 w' - P\frac{\partial^2\varphi}{\partial x^2} + Q\frac{\partial^2 w'}{\partial x^2} = 0\,, \quad \ldots \quad (11)$$

$$\text{where } R = \frac{(1-\nu^2)}{b\rho}\left(\frac{b}{\pi}\right)^2,\ P = \frac{3b}{\rho h^2}\left(\frac{b}{\pi}\right)^2,\ Q = \frac{2hp}{D}\left(\frac{b}{\pi}\right)^2. \quad \ldots \quad (12)$$

Two methods of development are now available. The first is to solve directly for u'; v', w' from equations (4) to (6); the second is to solve for w' and φ from (10) and (11), and then for u' and v' from (7) to (9). To decide on which is the better method depends on the boundary conditions to be satisfied in the particular case considered. Both types of solution are illustrated in what follows, as the case in which the edges are simply supported is dealt with by the first method, and the case in which the edges are fixed by the second.

Case in which Edges are Simply Supported.—When the strip buckles it is assumed in this case that any displacement of the edges is prevented, but that edges are free from couples. Expressed analytically, the boundary conditions are accordingly

$$\nu\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} = 0\,,\ w' = 0, \quad \ldots \quad (13)$$

$$u' = 0,\ v' = 0\,,$$

$$\text{when } y \text{ is } \pm\frac{\pi}{2}.$$

After observing that (4), (5) and (6) possess particular solutions of the form

$$u' = Ae^{\tau y}\sin mx,$$

$$v' = Be^{\tau y}\cos mx,$$

$$w' = Ce^{\tau y}\cos mx,$$

where τ and m are related by the equation

$$(\tau^2 - m^2)^4 - Qm^2(\tau^2 - m^2)^2 + PRm^4 = 0\,, \quad \ldots \quad (14)$$

we try for a complete solution in the form

$$
\begin{aligned}
u' &= \left\{\sum_{r=1}^{4} A_{mr} \cosh \tau_{mr} y\right\} \sin mx, \\
v' &= \left\{\sum_{r=1}^{4} B_{mr} \sinh \tau_{mr} y\right\} \cos mx, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (15) \\
w' &= \left\{\sum_{r=1}^{4} C_{mr} \cosh \tau_{mr} y\right\} \cos mx.
\end{aligned}
$$

Here the relations between the A's, B's and C's are found by direct substitution in any two of the equations (4) to (6), and the τ's are the roots of (14).

Substituting for u', v' and w' in (13), the boundary conditions take the form

$$
\begin{aligned}
&\sum_{r=1}^{4} C_{mr} \cosh \tau_{mr} \frac{\pi}{2} = 0, \\
&\sum_{r=1}^{4} C_{mr} (\tau_{mr}{}^2 - \nu m^2) \cosh \tau_{mr} \frac{\pi}{2} = 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (16) \\
&\sum_{r=1}^{4} A_{mr} \cosh \tau_{mr} \frac{\pi}{2} = 0, \\
&\sum_{r=1}^{4} B_{mr} \sinh \tau_{mr} \frac{\pi}{2} = 0.
\end{aligned}
$$

Then on substituting for u', v' and w', in (4) and (5) it follows that

$$
\frac{A_{mr}}{\Delta_A} = \frac{B_{mr}}{\Delta_B} = \frac{C_{mr}}{\Delta_C}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (17)
$$

where

$$
\begin{aligned}
\Delta_A &= \frac{bm}{2\pi\rho} (1-\nu) \left(\tau^2_{mr} + \nu m^2\right), \\
\Delta_B &= \frac{b\tau_{mr}}{2\pi\rho} (1-\nu) \left\{\left(\tau^2_{mr} - m^2\right) - (1+\nu) m^2\right\}, \\
\Delta_C &= \frac{(1-\nu)}{2} \left(\tau^2_{mr} - m^2\right)^2.
\end{aligned}
$$

Eliminating the A's, B's and C's from the equations (16) by means of (17), the condition that the boundary conditions should be consistent is

$$
| \, a_1 \; b_2 \; c_3 \; d_4 \, | = 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (18)
$$

where

$$
\begin{aligned}
a_r &= \left(\tau^2_{mr} - m^2\right), \\
b_r &= \left(\tau^2_{mr} - m^2\right)^2 \left(\tau^2_{mr} - \nu m^2\right), \\
c_r &= \left(\tau^2_{mr} + \nu m^2\right), \\
d_r &= \tau_{mr} \left[\left(\tau^2_{mr} - m^2\right) - (1+\nu) m^2\right] \tanh \tau_{mr} \frac{\pi}{2}.
\end{aligned}
$$

Equation (18) is the fundamental relation connecting up the buckling stress p, the wave length of the buckles $2b/m$, and the dimensions of the strip, and after simplification it can be written in the form

$$(1-\mu)^{\frac{3}{2}}\left[\tau_{m1}\left\{(1+\mu)^{\frac{1}{2}}-m\varkappa\,(1+\nu)\right\}^2\tanh\tau_{m1}\frac{\pi}{2}-\tau_{m2}\left\{(1+\mu)^{\frac{1}{2}}+m\varkappa\,(1+\nu)\right\}^2\tanh\tau_{m2}\frac{\pi}{2}\right]$$

$$=(1+\mu)^{\frac{3}{2}}\left[\tau_{m3}\left\{(1-\mu)^{\frac{1}{2}}-m\varkappa\,(1+\nu)\right\}^2\tanh\tau_{m3}\frac{\pi}{2}-\tau_{m4}\left\{(1-\mu)^{\frac{1}{2}}+m\varkappa\,(1+\nu)\right\}^2\tanh\tau_{m4}\frac{\pi}{2}\right] \quad (19)$$

where $\varkappa$ and μ are defined by the equations

$$\frac{p}{E}=\frac{2\pi^2h^2}{3\,(1-\nu^2)\,b^2\varkappa^2},$$

$$\frac{(1-\mu^2)}{\varkappa^4}=\frac{3\,(1-\nu^2)\,b^4}{\pi^4\,\rho^2\mathrm{h}^2},$$

and the τ's are expressed in terms of $\varkappa$ and μ by the relations

$$\tau_{m1}^2=m^2+\frac{m}{\varkappa}(1+\mu)^{\frac{1}{2}},$$

$$\tau_{m2}^2=m^2-\frac{m}{\varkappa}(1+\mu)^{\frac{1}{2}},$$

$$\tau_{m3}^2=m^2+\frac{m}{\varkappa}(1-\mu)^{\frac{1}{2}},$$

$$\tau_{m4}^2=m^2-\frac{m}{\varkappa}(1-\mu)^{\frac{1}{2}}.$$

It is immaterial if any of the τ's are purely imaginary, since in the above equation they only occur to even powers. For any given curvature, equation (19) gives p in terms of m. For some value of m, p will, however, be a minimum, and it is this p and the corresponding m which are the values required and which are plotted in Figs. 1 and 2.

*Case in which Edges are Fixed.**—In this case it is assumed that, when the strip buckles, the edges are completely fixed. After transferring the origin to the middle of one of the sides, the boundary conditions, when expressed analytically, are

$$\frac{\partial w'}{\partial y}=0,\ w'=0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

$$u'=0,\ v'=0,$$

when y is 0 or π; and the procedure is now to solve for w' and φ from (10) and (11).

We start by expressing w' in the form

$$w'=\left\{\sum_{r=1}^{\infty}A_r\sin ry+Ey+Fy^2+Gy^3\right\}\cos mx,$$

as this is the simplest form which is general and which at the same time can be differentiated term by term four times. The A's are arbitrary, and E, F and G, which are expressed in terms of the A's by means of the boundary conditions (20), are accordingly given by

$$E=-\sum_{r=1}^{\infty}rA_r,\ F=\frac{1}{\pi}\sum_{r=1}^{\infty}\left\{2+(-)^r\right\}rA_r,\ G=\frac{-1}{\pi^2}\sum_{r=1}^{\infty}\left\{1+(-)^r\right\}rA_r.$$

* For a detailed investigation of a somewhat similar method, see the author's paper The Elastic Stability of a Long and Slightly Bent Rectangular Plate under Uniform Shear. Proc. Roy. Soc. A. Vol. 162, page 62, 1937.

After substituting for w' in (10), we get an ordinary differential equation for φ which has for its general solution

$$\varphi = \cos mx\left[\left\{A\cosh my + B\sinh my\right\} + y\left\{C\cosh my + D\sinh my\right\}\right.$$

$$\left. + m^2 R\left\{\sum_{r=1}^{\infty}\frac{A_r \sin ry}{(r^2+m^2)^2} + \frac{1}{m^4}\left[\frac{4F}{m^2} + y\left(E + \frac{12G}{m^2}\right) + Fy^2 + Gy^3\right]\right\}\right].$$

The constants A, B, C, D are arbitrary, and are determined from the boundary conditions for u' and v', after these have been expressed in terms of w' and φ from the equations (7) to (9).

Having now obtained expressions for w' and φ which satisfy all the fundamental equations and boundary conditions except the equation (11), it remains to substitute for w' and φ in that equation. Doing this gives

$$\sum_{r=1}^{\infty} A_r (r^2 + m^2)^2 \sin ry + \left\{m^4(Ey + Fy^2 + Gy^3) - 2m^2\,(2F + 6Gy)\right\}$$

$$-m^2 Q\left\{\sum_{r=1}^{\infty} A_r \sin ry + Ey + Fy^2 + Gy^3\right\} + PRm^4 \sum_{r=1}^{\infty}\frac{A_r \sin ry}{(r^2+m^2)^2}$$

$$+Pm^2\left\{A\cosh my + B\sinh my + y\,(C\cosh my + D\sinh my)\right\}$$

$$+ PR\left\{\frac{4F}{m^2} + y\,\left((E + \frac{12G}{m^2}\right) + Fy^2 + Gy^3\right\} = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

which must be valid for all y in $(0, \pi)$. On expressing the entire left-hand side of (21) in terms of sines of multiples of y, it follows that the coefficient of $\sin ry$ ($r = 1, 2, - \infty$,) must vanish, since (21) can be regarded as an identity in y. After considerable algebraic reduction we accordingly deduce that

$$(r^2 + m^2)^2 A_r + m^4 \sum_{n=1}^{\infty} e_m A_n - 4m^2 \sum_{n=1}^{\infty} f_m A_n - m^2 Q A_r - m^2 Q \sum_{n=1}^{\infty} e_m A_n + \frac{PRm^4}{(r^2+m^2)^2} A_r$$

$$+ PR\left[L_r\left\{\sum_{n=1}^{\infty} A_n\,(\alpha_n + \gamma_n)\right\} + M_r\left\{\sum_{n=1}^{\infty}(-)^n A_n\,(\alpha_n + \gamma_n)\right\} + U_r \sum_{n=1}^{\infty} A_n\,\mu_n - V_r \sum_{n=1}^{\infty}\,(-)^n A_n \mu_n\right]$$

$$+ PR\left[\sum_{n=1}^{\infty} e_m\,A_n + \frac{4}{m^2}\sum_{n=1}^{\infty} f_m\,A_n\right] = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

where

$$\alpha_r = \frac{r^3 + (2+\nu)\, rm^2}{(r^2+m^2)^2}\,,\ \mu_r = -\,2\,(1+2\nu)\left\{2 + (-)^r\right\}\frac{r}{\pi m^3},$$

$$\gamma_r = -\,(2+\nu)\,\frac{r}{m^2} - 6\,(3+2\nu)\left\{1 + (-)^r\right\}\frac{r}{\pi^2 m^4}\,,$$

$$e_{rn} = -\,\frac{2}{r^2}\,f_{rn} = -\frac{4}{\pi^2 r^3}\left\{1-(-)^r\right\}\left\{2+(-)^n\right\}n - (-)^r\,\frac{12}{\pi^2 r^3}\left\{1+(-)^n\right\}n\,,$$

$$\mathrm{L}_r = (-)^r\,\mathrm{M}_r = \frac{4m\alpha_r}{\pi\,(1+\nu)}\ \frac{\mathrm{X} - \mathrm{Y}\cos r\pi}{\mathrm{Z}}\,,$$

$$\mathrm{U}_r = (-)^{r+1}\mathrm{V}_r = \frac{2nr}{\pi\,(1+\nu)\,\mathrm{Z}}\left[\frac{2m\pi(1+\nu)\cos r\pi\sinh m\pi}{(r^2+m^2)}\left\{1 + \frac{m^2\,(1+\nu)}{(r^2+m^2)}\right\}\right.$$

$$\left. +\ \frac{(3-\nu)\,(1-\nu)\sinh^2 m\pi - m^2\pi^2(1+\nu)^2}{(r^2+m^2)} - \frac{2m^2(1+\nu)\,(3-\nu)\sinh^2 m\pi}{(r^2+m^2)^2}\right], \quad (23)$$

and X, Y, Z are given by

$$\mathrm{X} = (3-\nu)\cosh m\pi\sinh m\pi + m\pi\,(1+\nu)\,,$$
$$\mathrm{Y} = (3-\nu)\sinh m\pi + m\pi\,(1+\nu)\cosh m\pi\,,$$
$$\mathrm{Z} = (3-\nu)^2\sinh^2 m\pi - m^2\pi^2\,(1+\nu)^2\,.$$

The equations (22) represent an infinite system of equations linear in the A's whose only solution is in general that in which all the A's are zero. If, however, the infinite determinant formed by eliminating the A's vanishes, the position is altered and there exists an infinite set of non-zero solutions. Since these involve non-zero values for w', they are positions of neutral equilibrium, and we have accordingly found an equation from which to deduce the critical values of Q, and hence of p. Expressing the fact that the infinite determinant vanishes, this equation is

$$\begin{vmatrix} \Omega_1 & 0 & \mathrm{N}_{13} & 0 & \mathrm{N}_{15} & 0 & \cdot\cdot & \cdot\cdot \\ 0 & \Omega_2 & 0 & \mathrm{N}_{24} & 0 & \mathrm{N}_{26} & \cdot\cdot & \cdot\cdot \\ \mathrm{N}_{31} & 0 & \Omega_3 & 0 & \mathrm{N}_{35} & 0 & \cdot\cdot & \cdot\cdot \\ 0 & \mathrm{N}_{42} & 0 & \Omega_4 & 0 & \mathrm{N}_{46} & \cdot\cdot & \cdot\cdot \\ \mathrm{N}_{51} & 0 & \mathrm{N}_{53} & 0 & \Omega_5 & 0 & \cdot\cdot & \cdot\cdot \\ 0 & \mathrm{N}_{62} & 0 & \mathrm{N}_{64} & 0 & \Omega_6 & \cdot\cdot & \cdot\cdot \end{vmatrix} = 0\,, \quad (24)$$

where

$$\Omega_r = (r^2+m^2)^2 + m^4 e_{rr} - 4m^2 f_{rr} + \frac{\mathrm{PR}m^4}{(r^2+m^2)^2} - m^2\mathrm{Q}\,(1+e_{rr})$$

$$+\ \mathrm{PR}\left\{e_{rr} + \frac{4}{m^2}f_{rr} + (\alpha_r+\gamma_r)\,\{\mathrm{L}_r + (-)^r\,\mathrm{M}_r\} + \mu_r\,\{\mathrm{U}_r - (-)^r\,\mathrm{V}_r\}\right\}; \quad (25)$$

$$\mathrm{N}_{ij} = m^4 e_{ij} - 4m^2 f_{ij} - m^2\mathrm{Q}e_{ij}$$

$$+\ \mathrm{PR}\left\{e_{ij} + \frac{4f_{ij}}{m^2} + (\alpha_j+\gamma_j)\,\{\mathrm{L}_i + (-)^j\,\mathrm{M}_i\} + \mu_j\{\mathrm{U}_i - (-)^j\,\mathrm{V}_i\}\right\}. \quad (26)$$

A proof will now be given that the determinant in (24) is convergent, as otherwise the solution given by that equation is merely a formal one. From (23) we deduce that the orders of magnitude of the various quantities involved are

$$\begin{aligned} \alpha_r &= 0\left(\frac{1}{r}\right), \gamma_r = 0(r), \\ L_r &= 0\left(\frac{1}{r}\right), \mu_r = 0(r), \\ M_r &= 0\left(\frac{1}{r}\right), e_m = 0\left(\frac{n}{r^3}\right), \\ U_r &= 0\left(\frac{1}{r}\right), f_m = 0\left(\frac{n}{r}\right). \\ V_r &= 0\left(\frac{1}{r}\right), \end{aligned} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

Now divide the $(2r - 1)$th and $2r$th rows by r, and the $(2r - 1)$th and $2r$th columns by r^3, then from (25) to (27) it follows that the product of the diagonal terms, and the sum of the non-diagonal terms are each absolutely convergent, and hence that the determinant itself converges.*

Since equation (24) is of indefinitely large degree in Q, it will have an infinite number of roots, but it is, of course, only with the smallest of these roots that we are concerned in practice. Owing to the form of the determinant in (24), the approximate determinant formed by taking its first $2n$ rows and columns can be expressed as the product of two determinants each of order n. The equation (24) can hence be written in the form

$$\begin{vmatrix} \Omega_1 & N_{13} & N_{15} & \ldots \\ N_{31} & \Omega_3 & N_{35} & \ldots \\ N_{51} & N_{53} & \Omega_5 & \ldots \end{vmatrix} \begin{vmatrix} \Omega_2 & N_{24} & N_{26} & \ldots \\ N_{42} & \Omega_4 & N_{46} & \ldots \\ N_{62} & N_{64} & \Omega_6 & \ldots \end{vmatrix} \ldots = 0, \qquad (28)$$

and it remains to obtain successive approximations to each of these two infinite determinants. A very little calculation is sufficient to show that the smallest value of Q, and hence of p, is given by the determinant on the left in equation (28), and hence the vanishing of this determinant is the fundamental equation connecting up the buckling stress p, the wave length of the buckles $2b/m$, and the dimensions of the strip. The convergence of the determinant is extremely rapid, and over the range of values considered, the second order determinant formed from the first two rows and columns gives results which are accurate to 1 per cent.

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* Whittaker and Watson, Modern Analysis (4th ed.) Chap. 2, § 2, 8.

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