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A Non-Axisymmetric End-Wall Boundary-Layer Theory for Axial Compressor Rows

by

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SUMMARY

A second equation is deduced for the growth of the end-wall boundary layer which connects the work of the mean blade force along the mean velocity with the energy of the secondary disturbances in the boundary layer. Secondary flow theory is used to obtain two equations for two boundary-layer parameters. The method is regarded as an addition and modification to the Mellor and Wood treatment.

*Replaces A.R.C.35 863

CONTENTS

1. Introduction

It is generally agreed that the problem of the annulus wall boundary layer growth in an axial compressor is at the same time a problem in the correct representation of the variation of blade forces through the layer. As Mellor and Wood (1970) point out, force defect cannot be neglected in a meaningful compressor theory. In that work a useful advance was made by them with the consideration that the force defect could not be described at the outset but could be eliminated by the statement that the force defect was normal to the blade surface together with the assumption that the relative flow at any row exit was "collateral", i.e. it had a variation across the layer such that the flow angle stays constant and equal to the mainstream value. Departures from this condition were considered, for example, as being due to the influence of tip clearance in the case of flow past blade tips.

In deriving their equations, Mellor and Wood used pitchwise averaging of the equations of motion (along the tangential, z direction, x being the axial and y the spanwise direction, normal to the annulus wall). In deducing these equations a number of terms made their appearance in both the x- and z-direction momentum equations which they subsequently neglected. These terms which represent the contribution to mean momentum due to the pitchwise variation oflocal velocity components combined in product form may, for short, be called (with sign reversed) apparent stresses. The neglect of these terms in the context of the annulus wall boundary layer, even in the case of closely-pitched blades is questionable.

An attempt to assess the influence of these apparent stresses in the simplest possible situation, (that of two-dimensional blade-to-blade flow) in the hope of gaining some insight to the overall influence of some of these terms, was made by Railly (1973). In that treatment the averaging process was applied to both the momentum and the energy equation and it was shown that the condition of normality of blade force vector to mean relative velocity in the context of an averaged treatment of the blade row had to be relaxed. In an earlier discussion Horlock and Marsh (1971) had shown, for the three-dimensional flow in a blade row, that if the apparent stresses were not negligible then there was no mean force flow that could simultaneously satisfy both momentum continuity and normality.

An application of the averaged energy equation to the annulus wall boundary layer to predict the departure from the collateral condition at blade row exit was made by Railly (1974); however there were too many simplifications in that treatment for accurate conclusions to be drawn. Measurements of the apparent stresses at exit from a single rotor row were made, using hot-wire anemometry, by Ball and Roy (1973) and the magnitudes of the apparent stresses across the hub wall boundary layer were shown to be appreciable implying a considerable departure from the collateral condition there. Although the blade row in question was rather highly loaded there is enough support for the view that a proper treatment of the annulus wall boundary layer should include the influence of the apparent stresses.

In the absence of the apparent stresses, the flow ought to be collateral, not only at exit, but throughout the blade row, since closely-pitched blading could permit no other. For this reason it is proposed to dispense, formally, with the condition of collateral flow at exit and to allow the mean velocity vector some 'latitude'. This, of course, is the secondary flow situation where initial departure from the collateral condition is equivalent to the presence of upstream streamwise vorticity.

The object of the following treatment, therefore, is to develop an equation for the departure from collateral flow in terms only of the unknown 'apparent stresses' which also make their appearance in the equations of motion. These stresses may then be evaluated in terms of secondary flow theory and the system may be solved in terms of two boundary-layer parameters. This second equation will be deduced from a statement of the averaged energy.

2. The Energy-Dissipation Equation

The use of the term 'dissipative' is to some extent misleading since genuine energy dissipation into heat is not being considered but rather a conversion of the directed energy of an undisturbed flow into the energy of the disturbance velocities defined by quantities such as u', v', w' given by

$$\begin{aligned} u' &= u - \bar{u} \\ v' &= v - \bar{v} \\ w' &= w - \bar{w} \end{aligned} \qquad \ldots(1)$$

where $\bar{u}, \bar{v}, \bar{w}$ are pitchwise averages and u, v, w are time-steady local velocities.

In fact it is proposed that the actual dissipative mechanism of fluid friction can be dismissed by virtue of the following statement: That the work of the average of the viscous and turbulent shear forces on the mean velocity field $(\bar{u}, \bar{v}, \bar{w})$ is about equal to the work of these forces on the actual field (u, v, w). If then we derive an equation for the balance of mechanical energy for the actual flow field and subtract from it the equation for mechanical energy of the mean velocity field (as described by the averaged equations of motion) we obtain an equation for the dissipation of energy attached to the disturbance field, u', v', w'.

The energy for the actual flow field is simply

$$(\underline{q}.\nabla)\ E = 0 \qquad \ldots(2)$$

where $E = p/\rho + \frac{1}{2}(u^2 + v^2 + w^2)$ in incompressible flow. By virtue of the equation of continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \ldots(3)$$

then

$$u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = \frac{\partial}{\partial u}(pu) + \frac{\partial}{\partial y}(pv) + \frac{\partial}{\partial z}(pv) \qquad \ldots(4)$$

Pitch-averaging the R.H.S. of equation (4) it follows that the only non-zero terms are

$$\frac{\partial}{\partial x}(\overline{pu}) + \frac{\partial}{\partial x}(\overline{p'u'}) + \frac{\partial}{\partial y}(\overline{pv}) + \frac{\partial}{\partial y}(\overline{p'v'})$$

provided it is assumed that the variation of blade angle with y is zero. Referring to the k.e. terms appearing in equation (2), then

u/

$$u\frac{\partial}{\partial x}(q^{2}/2) + v\frac{\partial}{\partial y}(q^{2}/2) + w\frac{\partial}{\partial z}(q^{2}/2) = \frac{\partial}{\partial x}\left(\frac{u^3}{2}\right)$$

$$+\frac{\partial}{\partial x}\left(\frac{uv^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{uw^2}{2}\right) + \frac{\partial}{\partial y}\left(\frac{vu^2}{2}\right) + \frac{\partial}{\partial y}\left(\frac{v^3}{2}\right) + \frac{\partial}{\partial y}\left(\frac{vw^2}{2}\right)$$

$$+\frac{\partial}{\partial z}\left(\frac{wu^2}{2}\right) + \frac{\partial}{\partial z}\left(\frac{wv^2}{2}\right) + \frac{\partial}{\partial z}\left(\frac{w^3}{2}\right) - q^{2}/2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\ldots(5)$$

making the substitutions from equations (1) and carrying out the pitchwise, z, averaging then, considering the first three terms on the R.H.S. of equation (5), they become,

$$\frac{\partial}{\partial x}\int_a^b u(q^{2}/2)\,dz - (uq^{2}/2)_2 \tan\beta + (uq^{2}/2)_1 \tan\beta \qquad \ldots(6)$$

with $\tan\beta = db/dx = da/dx$, where a,b are successive z-co-ordinates of the (thin) blade surfaces. Treating the integral, it follows that the average is

$$\bar{u}\left(\frac{\overline{u'^2}}{2} + \frac{\overline{v'^2}}{2} + \frac{\overline{w'^2}}{2}\right) + \bar{u}\left(\frac{\bar{u}^2}{2} + \frac{\bar{v}^2}{2} + \frac{\bar{w}^2}{2}\right) + \bar{u}\,\overline{u'^2} +$$

$$\bar{v}\,\overline{u'v'} + \bar{w}\,\overline{u'w'} \qquad \ldots(7)$$

(dropping two terms of 3rd order in the disturbance velocities). Similarly, the second three terms of (5) become

$$\bar{v}\left(\frac{\bar{u}^2}{2} + \frac{\bar{v}^2}{2} + \frac{\bar{w}^2}{2}\right) + \bar{v}\,k + \bar{u}\,\overline{u'v'} + \bar{v}\,\overline{v'^2} + \bar{w}\,\overline{v'w'} \qquad \ldots(8)$$

using k for disturbance energy, $(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})/2$.

Averaging the last three terms of equation (5) these are seen to cancel exactly the last two terms of expression (6).

The final energy equation is then

$$\frac{\partial}{\partial x}(\overline{pu}) + \frac{\partial}{\partial y}(\overline{pv}) + \rho\frac{\partial}{\partial x}[\bar{u}(\bar{u}^2 + \bar{v}^2 + \bar{w}^2)/2] + \rho\frac{\partial}{\partial y}[\bar{v}(\bar{u}^2 + \bar{v}^2 +$$

$$\bar{w}^2)/2] + \frac{\partial}{\partial x}(\overline{p'u'}) + \frac{\partial}{\partial y}(\overline{p'v'}) + \rho\frac{\partial}{\partial x}[\bar{u}k + \bar{u}\,\overline{u'^2} + \bar{v}\,\overline{u'v'} + \bar{w}\,\overline{u'w'}]$$

$$+ \rho\frac{\partial}{\partial y}[\bar{v}\,k + \bar{u}\,\overline{u'v'} + \bar{v}\,\overline{u'^2} + \bar{w}\,\overline{v'w'}] = 0 \qquad \ldots(9)$$

Turning/

Turning now to the averaged equations of motion, it may be shown (cf. Mellor and Wood, (loc. cit.)) that these are

$$\rho \bar{u} \frac{\partial \bar{u}}{\partial x} + \rho \bar{v} \frac{\partial \bar{u}}{\partial y} = - \frac{\partial \bar{p}}{\partial x} + \rho F_x - \frac{\partial}{\partial x} (\rho \overline{u'^2}) - \frac{\partial}{\partial y} (\rho \overline{u'v'}) \qquad \ldots(10a)$$

$$\rho \bar{u} \frac{\partial}{\partial x} (\bar{v}) + \rho \bar{v} \frac{\partial}{\partial y} (\bar{v}) = - \frac{\partial \bar{p}}{\partial y} - \rho \frac{\partial}{\partial x} (\overline{v'u'}) - \rho \frac{\partial}{\partial y} (\overline{v'^2}) \qquad \ldots(10b)$$

$$\rho \bar{u} \frac{\partial \bar{w}}{\partial x} + \rho \bar{v} \frac{\partial \bar{w}}{\partial y} = \rho F_z - \frac{\partial}{\partial x} (\rho \overline{u'w'}) - \frac{\partial}{\partial y} (\rho \overline{v'w'}) \qquad \ldots(10c)$$

in which, for the reasons stated above, the viscous and turbulent stresses have been left out.

Multiplying equation (10a) by $\bar{u}$, (10b) by $\bar{v}$ and (10c) by $\bar{w}$ and adding then

$$\rho \bar{u}^2 \frac{\partial \bar{u}}{\partial x} + \rho \bar{u} \bar{v} \frac{\partial \bar{u}}{\partial y} + \rho \bar{u} \bar{w} \frac{\partial \bar{w}}{\partial x} + \rho \bar{v} \bar{w} \frac{\partial \bar{w}}{\partial y} = - \bar{u} \frac{\partial \bar{p}}{\partial x}$$

$$+ \bar{u} F_x + \bar{w} F_z - \rho \bar{u} \frac{\partial}{\partial y} (\overline{u'^2}) - \rho \bar{u} \frac{\partial}{\partial y} (\overline{u'v'}) - \bar{w} \frac{\partial}{\partial x} (\rho \overline{u'w'})$$

$$- \bar{w} \frac{\partial}{\partial y} (\rho \overline{w'v'}) \qquad \ldots(11)$$

Subtracting equation (9) from equation (11) eliminates the terms containing mean pressure and transposing it follows that:

$$- \bar{u} F_x - \bar{w} F_z = + \bar{u} \frac{\partial}{\partial x} k + \bar{v} \frac{\partial k}{\partial y} + (\overline{u'^2} - \overline{v'^2}) \frac{\partial \bar{u}}{\partial x} + \overline{u'v'}$$

$$\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) + \overline{u'w'} \frac{\partial \bar{w}}{\partial x} + \overline{v'w'} \frac{\partial \bar{w}}{\partial y} + \frac{\partial}{\partial x} (\overline{p'u'}/\rho) + \frac{\partial}{\partial y} (\overline{p'v'}/\rho) \qquad \ldots(12)$$

in which use has been made of the averaged equation of continuity,

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \qquad \ldots(13)$$

3. Boundary Layer Approximation

In the context of the permissible approximations in the annulus wall boundary layer, equation (12) may be simplified. Consideration may first be given to the pressure-velocity product terms on the right. In general we expect δp across the boundary-layer thickness, h, to be $O(h)$ hence it is reasonable approximation to assume that throughout the layer

$$p_b - p_a \doteq (p_b - p_a)_e = -\rho \frac{F_{xe}}{s} \qquad \ldots(14)$$

This assumption cannot be allowed throughout the analysis for it would reduce to the earlier assumptions of constant blade force; in the evaluation of one term of equation (12) it may be permitted on the grounds that the variation is small. Thus, assuming a linear variation of pressure between blades,

$$p' = (p_b - p_a)\ 2z/s$$

giving

$$\overline{p'u'} = -2\rho F_{xe}\ \overline{zu'} \qquad \ldots(15)$$

The second term involving p' will not be considered as later it will be seen not to be needed.

Turning to the momentum equations equations (10), these may be treated in the conventional manner except that equation (10b) must be examined as to the order of magnitude of the second and third terms on the right. If v' is of the same order as u' it follows that the third term is $O(|u'|.|u'|/h)$ while the second is $O(|u'|.|u'|)$ so that the latter may be neglected. Equation (10b) then reduces to

$$\rho \frac{\partial}{\partial y}(\overline{v'^2}) = -\frac{\partial p}{\partial y} \qquad \ldots(16)$$

It is not proposed to attempt a solution of the system of differential equations but to develop integral relationships by integrating equations (10a), (10c), (12) and (16) across the boundary layer.

Starting with equation (16), if it be true that $\overline{v'^2}$ is zero at the edge of the layer, $y = h$, since it is also zero at the wall then

$$p_e = p(h) = p(o) \qquad \ldots(17)$$

and, in addition, integrating to a point y,

$$\rho\overline{v'^2} = -p(y) + p(o)$$

which after differentiation w.r.t. x, gives

$$\frac{\partial}{\partial x}(\rho\overline{v'^2}) = -\frac{\partial p}{\partial x} + \frac{\partial p_e}{\partial x} \qquad \ldots(18)$$

Substituting/

Substituting equation (18) into equation (10a) and carrying out the integration of equation (10a) in the usual manner (also of equation (10c)) it follows that (cf. Mellor and Wood (loc. cit.)),

$$\frac{d}{dx}(u_e^2\,\theta_x) + \delta_x^* u_e \frac{du_e}{dx} = \frac{\tau_{ox}}{\rho} + \int_o^h (F_{xe} - F_x)\,dy + \frac{\partial}{\partial x}\int_o^h (\overline{u'^2} - \overline{v'^2})\,dy \qquad \ldots(19)$$

$$\frac{d}{dx}(u_e w_e \theta_z) + \delta_x^* u_e \frac{dw_e}{dx} = \frac{\tau_{oz}}{\rho} + \int_o^h (F_{ze} - F_z)\,dy + \frac{\partial}{\partial x}\int_o^h \overline{u'w'}\,dy \qquad \ldots(20)$$

again making use of the condition that $\overline{u'^2}$ etc. are zero at $y = h$.

Referring now to equation (12) it is necessary that this be integrated over the boundary-layer thickness also and the presence of the blade force terms, F_x and F_z, would seem to present a difficulty. However, once again the assumption is made that the departures of these from the values F_{xe}, F_{ze} are slight. Making, now, use of the condition that blade force must be normal to the blade surface, this is simply

$$F_x + F_z \tan\beta = 0 \qquad \ldots(21)$$

Hence the term on the left of equation (12) reduces to

$$F_z(\bar{u}\tan\beta - \bar{w}) \simeq F_{ze}(\bar{u}\tan\beta - \bar{w})$$

$$= F_{ze} w_e[(1-\bar{w}/w_e) - (1-\bar{u}/u_e)] \qquad \ldots(22)$$

Integration of this result over the thickness, h, gives

$$w_e\,F_{ze}(\delta_z^* - \delta_x^*)$$

so that the R.H.S. of equation (12), after integration, is proportional to the difference in displacement thicknesses of the tangential and axial profiles, i.e., to the departure from the collateral condition. Applying the condition of normality, equation (21), to equations (19) and (20) allows the complete elimination of the force deficit integrals by multiplying equation (20) by $\tan\beta$ and adding this to equation (19). The result is the first of the integral equations of the annulus boundary layer as follows:

$$\frac{d}{dx}$$

$$\frac{d}{dx}(u_e^2\ \theta_x) + \tan\beta\frac{d}{dx}(u_e\ w_e\ \theta_z) + \delta_x^*\left(u_e\ \frac{du_e}{dx} + u_e\ \tan\beta\frac{dw_e}{dx}\right)$$

$$= (\tau_{ox} + \tau_{oz}\ \tan\beta)/\rho\ = \frac{\partial}{\partial x}\int_o^h(\overline{u'^2} - \overline{v'^2})\ dy\ +$$

$$\tan\beta\frac{\partial}{\partial x}\int_o^h \overline{u'w'}\ dy \qquad \ldots(23)$$

The second boundary-layer equation comes by integrating equation (12) and this may be carried out since the stress terms may all be assumed to be zero at the edge. The last term on the right may be seen to disappear on integration and the last but one, from equation (15), is simply

$$-2\int_o^h \frac{\partial}{\partial x}\left(F_{xe}\ \overline{zu'}\right) dy\ = -2\int_o^h \frac{\partial}{\partial x}\left(\frac{F'_{xe}\ u_e^3}{\sigma}\left(\frac{\overline{zu'}}{su_e}\right)\right) dy \qquad \ldots(24)$$

where F'_{xe} is a non-dimensionalised form of F_{xe} and σ is axial solidity, L/S. Evaluation of the integral in suitable form is important for similar integrals will emerge in the treatment of the other terms. We assume that the function in brackets inside the above integral on the right after being normalised by division of s and u_e is a similar function dependent only on a parameter, α, as follows:

$$\left(\frac{\overline{zu'}}{su_e}\right) = F(\alpha,\ \eta) \qquad \ldots(25)$$

where α is some property of the mean profile, as yet unspecified, and η is y/h. Hence the integral in equation (24), after carrying out the differentiation w.r.t. x, noting that

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial \alpha}\ \frac{d\alpha}{dx} - \frac{\eta}{h}\ \frac{dh}{dx}\ \frac{\partial F}{\partial \eta}$$

may be shown to be given by

$$-\frac{2}{\sigma}\ u_e^3\left[h\ \frac{dF'_{xe}}{dx}\int_o^1 F d\eta - hF'_{xe}\ \frac{d\alpha}{dx}\ \frac{\partial}{\partial\alpha}\int_o^1 F d\eta - F'_{xe}\ \frac{dh}{dx}\int_o^1 F d\eta\right] \qquad \ldots(26)$$

integrating by parts to obtain the last term; the variation of u_e with x has been ignored for convenience. This has been assumed in the later development but no problems are involved if u_e has a variation with x, merely an increase in the number of terms.

Turning to the remaining terms of equation (12), the first two, by virtue of equation (13) may be replaced by

$$\int_o^h$$

$$\int_0^h \frac{\partial}{\partial x}(\overline{u}k)\, dy + \int_0^h \frac{\partial}{\partial y}(\overline{v}k)\, dy = \frac{\partial}{\partial x}\int_0^h \overline{u}\, k\, dy$$

since the second integral is zero. The remaining integral of these two may be written

$$\frac{\partial}{\partial x}\left(h\, u_e^3 \int_0^1 \frac{\overline{u}}{u_e}\, \frac{k}{u_e^2}\, d\eta\right) = u_e^3\, \frac{\partial}{\partial x}\left(h \int_0^1 G(\alpha,\eta)\, d\eta\right)$$

$$= u_e^3 \left(h \frac{d\alpha}{dx}\, \frac{\partial}{\partial \alpha}\int_0^1 G(\alpha,\eta)\, d\eta + \frac{dh}{dx}\int_0^1 G(\alpha,\eta)\, dn\right)$$

The third integral from the integration of equation (12) may be written

$$\int_0^h (\overline{u'^2} - \overline{v'^2})\, \frac{\partial \overline{u}}{\partial x}\, dy = h\, u_e^2 \int_0^1 \frac{\overline{u'^2} - \overline{v'^2}}{u_e^2}\, \frac{\partial}{\partial x}\left(u_e\, \frac{\overline{u}}{u_e}\right) d\eta$$

$$= h u_e^3 \frac{d\alpha}{dx}\int_0^1 H(\alpha,\eta)\, U_\alpha(\alpha,\eta)\, d\eta - u_e^3 \frac{dh}{dx}\int_0^1 H\eta\, U_\eta\, d\eta$$

the subscripts α and η in the above imply differentiation. The function H is $(\overline{u'^2} - \overline{v'^2})/u_e^2$ and $U = \overline{u}/u_e$. Again, variation of u_e with x has ignored.

The fifth integral is similar to the above and becomes

$$h u_e^2\, w_e\, \frac{d\alpha}{dx}\int_0^1 P(\alpha,\eta)\, W_\alpha(\alpha,\eta)\, d\eta = h u_e^2 \frac{dw_e}{dx}\int_0^1 P(\alpha,\eta)\, W(\alpha,\eta)\, d\eta$$

$$- u_e^2\, w_e\, \frac{dh}{dx}\int_0^1 P\eta\, W_\eta\, d\eta$$

where $P = \overline{u'w'}/u_e^2$ and $W = \overline{w}/w_e$.

Turning to the fourth integral from equation (12), the second term in the brackets is to be regarded as negligible compared with the first and this may be written

$$\int_0^h \overline{u'v'}\, \frac{\partial \overline{u}}{\partial y} = u_e^3 \int_0^1 Q(\alpha,\eta)\, U_\eta(\alpha,\eta)\, d\eta$$

where the subscript implies differentiation w.r.t. η and the meaning of Q and U is clear.

Finally/

Finally the sixth integral is

$$\int_0^h \overline{v'w'} \frac{\partial \overline{w}}{\partial y} \, dy = u_e^2 \, w_e \int_0^1 R(\alpha,\eta) \, W_\eta(\alpha,\eta) \, d\eta$$

and the meaning of functions R and W is clear.

Applying this procedure to the terms of the momentum integral equation, equation (23), the equation is transformed into a first-order differential equation in h and α. The functions θ_x, θ_z, δ_x^*, δ_z^* may all be written in terms of the functions U and W. Thus from their definitions,

$$\theta_x = \int_0^h \frac{\overline{u}}{u_e}\left(1 - \frac{\overline{u}}{u_e}\right) dy, \qquad \theta_z = \int_0^h \frac{\overline{u}}{u_e}\left(1 - \frac{\overline{w}}{w_e}\right) dy,$$

$$\delta_x^* = \int_0^h \left(1 - \frac{\overline{u}}{u_e}\right) dy, \qquad \delta_z^* = \int_0^h \left(1 - \frac{\overline{w}}{w_e}\right) dy$$

it may be shown that

$$\frac{\partial}{\partial x}\left(u_e^2 \, \theta_x\right) = u_e^2 \, h \frac{d\alpha}{dx} \int_0^1 U_\alpha (1 - 2U) \, d\eta - u_e^2 \frac{dh}{dx} \int_0^1 \eta \, U_\eta (1 - 2U) \, d\eta$$

$$\frac{\partial}{\partial x}\left(u_e \, w_e \, \theta_z\right) = u_e^2 \tan\beta \left[h \frac{d\alpha}{dx} \int_0^1 \left(1 - \frac{\partial}{\partial \alpha}(UW)\right) d\eta + \frac{dh}{dx} \int_0^1 U(1-W) \, d\eta \right]$$

$$+ u_e^2 \, h \frac{d}{dx}(\tan\beta) \int_0^1 U(1 - W) \, d\eta.$$

We also have the results

$$\frac{\partial}{\partial x} \int_0^h (\overline{u'^2} - \overline{v'^2}) \, dy = u_e^2 \frac{dh}{dx} \int_0^1 H \, d\eta + u_e^2 \, h \frac{d\alpha}{dx} \frac{\partial}{\partial \alpha} \int_0^1 H \, d\eta$$

$$\frac{\partial}{\partial x} \int_0^h \overline{u'w'} \, dy = u_e^2 \frac{dh}{dx} \int_0^1 P \, d\eta + u_e^2 \, h \frac{d\alpha}{dx} \frac{\partial}{\partial \alpha} \int_0^1 P \, d\eta$$

(where the functions P and H were introduced above) which may be substituted into equation (23)).

All of the above transformations of the integral quantities may be substituted into equations (12) and (23) and the resulting equations rearranged to yield the following pair of first-order ordinary differential equations in h and α, the independent variable being x:

$$A(1,1)\ \mathring{h} + A(1,2)\ \mathring{\alpha} = C(1) \qquad \ldots(23a)$$

$$A(2,1)\ \dot{h} + A(2,2)\ \dot{\alpha} = C(2) \qquad \ldots(12a)$$

where

$$A(1,1) = \int_0^1 (U - U^2 - H)\, d\eta + \tan^2\beta \int_0^1 U(1-W)\, d\eta - \tan\beta \int_0^1 P\, d\eta$$

$$A(1,2) = h \int_0^1 U_\alpha(1-2U)\, d\eta + h \tan^2\beta \int_0^1 \left(1 - \frac{\partial}{\partial\alpha}(UW)\right) d\eta - h \frac{\partial}{\partial\alpha} \int_0^1 (H + P\tan\beta)\, d\eta$$

$$C(1) = -c\, h \tan\beta \int_0^1 (1-UW)\, d\eta + (\tau ox + \tau oz)/\rho\, u_e^2$$

$$A(2,1) = \int_0^1 (G - H\eta\, U\eta - \tan\beta\, P_\eta\, W_\eta)\, d\eta + \frac{2}{\sigma} F'_{xe} \int_0^1 F\, d\eta$$

$$A(2,2) = h \frac{\partial}{\partial\alpha} \int_0^1 G\eta + h \int_0^1 H\, U_\alpha\, d\eta + h \tan\beta \int_0^1 PW_\alpha\, d\eta + \frac{2}{\sigma} F'_{xe}\, h \frac{\partial}{\partial\alpha} \int_0^1 F d\eta$$

$$C(2) = c\,h \tan\beta \int_0^1 (U-W)\, d\eta - \int_0^1 QU_\eta\, d\eta - hc \int_0^1 PW\, d\eta - \tan\beta \int_0^1 PW_\eta\, d\eta$$

$$+ \frac{2}{\sigma} h \frac{dF'_{xe}}{dx} \int_0^1 F d\eta$$

where $c = \frac{d}{dx}(\tan\beta)$ which may be regarded as a constant for the particular row; in that case the blade force variation may be equated to a constant.

These equations could be solved quite easily if all the functions could be written as analytic functions of η with parameter, α. It remains however to establish the way in which the various quantities depend upon η and α. Thus it is necessary to establish a causal link between a particular profile of one of the components of mean velocity on the one hand and, on the other hand, the remaining component of mean velocity and the disturbance components which give rise to the various terms involving $\overline{u'^2}$, $\overline{u'v'}$ etc.

For/

For this purpose we have no alternative than to introduce a secondary flow hypothesis into the discussion. In addition it is necessary to make allowance for the fact that near the wall the flow is influenced by viscous and turbulent shear.

With regard to the introduction of the secondary flow hypothesis into the problem, considerable help may be obtained from the structure proposed by Mellor and Wood (loc.cit.) for the secondary flow pattern and used by Horlock (1973) to form the basis of an approximate solution for the cross-flow.

4. Use of the Horlock Cross-Flow Profile

By starting with the Mellor and Wood secondary flow structure, which assumed v to depend linearly upon n (the co-ordinate normal to the channel flow direction, see Fig. 1, which is also the direction for collateral flow) and w_n to depend upon a quadratic function of n, Horlock showed that the cross-flow could be derived from streamwise vorticity provided the latter could be written in the form

$$\zeta_s = B(d\bar{V}c/dy)_{entry} \qquad \ldots(27)$$

here $\bar{V}_c$ stands for 'collateral' component of mean velocity (i.e. main streamwise component).

Now it may be shown, considering any row, for example a stator row, that provided the flow leaving the previous rotor exit is approximately collateral, then a form of equation (27) always obtains. Thus if, at rotor exit,

$$\zeta_x = \frac{\partial \bar{w}^{(r)}}{\partial y} = \tan \beta^{(r)} \frac{\partial \bar{u}}{\partial y}$$

$$\zeta_z = \frac{\partial \bar{u}}{\partial y}$$

are axial and tangential components of vorticity of the mean flow then the streamwise vorticity along the absolute free-stream direction is

$$\zeta_s = \frac{\partial \bar{u}}{\partial y}(\tan \beta^{(r)} \cos \beta - \sin \beta) \qquad \ldots(28a)$$

while the normal vorticity is

$$\zeta_n = -\frac{\partial \bar{u}}{\partial y}(\tan \beta^{(r)} \sin \beta + \cos \beta). \qquad \ldots(28b)$$

Using the simplified form of Hawthorne's vorticity expression (Horlock loc.cit it follows, at a later station "2", provided $\beta_2 - \beta_1$ is small, that

$$\zeta_{s2} = -\left\{ \frac{\cos \beta_1}{\cos \beta_2} \left[\sin \beta_1 - \cos \beta_1 \tan \beta_1^{(r)} \right] + 2(\beta_2 - \beta_1)(\cos \beta_1 + \tan \beta_1^{(r)} \sin \beta_1 \right\} \frac{\partial \bar{u}}{\partial y} \qquad \ldots(29)$$

and/

and since $\overline{V}_c = \overline{u} \sec \beta$, that equation (29) is a form of equation (27).

If equation (29) and subsequently the Horlock profile were used, then the system of (29), (23a) and (12a) would be solved in terms of a parameter, α, which could relate to the profile of V_c.

Assuming for the inviscid region of the flow that Horlock's approximate cross-flow profile is valid then, in the present nomenclature, equations (58) of his paper become

$$\left.\begin{aligned} \overline{w}_n &= B(V_e - \overline{V}_c) - 2\sqrt{3}\; B V_e \frac{\delta^*}{S'} e^{-ky}, \quad y < \delta \\ \overline{w}_n &= -2\sqrt{3}\, B V_e \frac{\delta^*}{S'} e^{-ky}, \quad y > \delta \end{aligned}\right\} \qquad \ldots(30)$$

where δ^* relates to the collateral profile and $k\delta = \sqrt{12}\,\dfrac{\delta}{S'}$. At this point we meet with a slight difficulty in that the outer limit of integration of the integrals of equations (23) and (12) is h where terms like $\overline{u'v'}$ etc. are sensibly zero, whereas δ is the edge of the profile of mean main streamwise velocity. It is essential that h be chosen in some simple manner that will correctly meet the above condition. For $y > \delta$ equation (30) may be written

$$\overline{w}_n(y) = \overline{w}_n(\delta)\, e^{k\delta(1-y/\delta)} \qquad \ldots(31)$$

The attenuation factor in this equation should have a (numerical) exponent in excess of 3 for $\overline{w}_n(y)$ to be small enough. This means that

$$y(\text{outer}) = \delta + \frac{3}{\sqrt{12}}\, S' \simeq \delta + S'$$

where the attenuation is now $\sqrt{12}$. Thus the outer limit is effectively large enough for a distance, S', greater than δ. Thus if we ignore the variation of S' with x, then $d\delta/dx$ equals dh/dx.

This result is strictly applicable to $k\delta$ small, but Horlock shows that it is quite good to values at least up to $\sqrt{3}$.

Utilising more of the secondary flow relationships:

$$v = v' = v_s(y)(n-n_m)/S' \qquad \ldots(32)$$

$$w_n = \frac{6\overline{w}_n(y)}{S'^2}\left[\frac{S'^2}{4} - (n - n_m)^2\right] \qquad \ldots(33)$$

and

$$\frac{dv_s}{dy} = \frac{12\overline{w}_n}{S'} \qquad \ldots(34)$$

The local axial velocity may be obtained from equation (33) provided it is assumed that the collateral velocity component of the total local velocity has no variation with z. Indeed simple secondary flow theory has nothing

to/

5. Choice of the α Parameter

All the functions appearing in equations (23a) and (12a) are dependent on η and contain α as a parameter. We are still at liberty to decide the role to be played by the latter. So far no mention has been made of the influence of friction upon either the streamwise profile or the cross-flow profile. Reference to the 'cross-flow' equation (equations (12) and (12a)) shows that no friction terms are involved. From inspection of the 'combined' momentum equation (equation (23)) it may be seen that the only friction component involves the linear combination of the two components of wall shear. Resolving the total shear stress in terms of the collateral and normal components:

$$\tau_{ox} + \tau_{oz} \tan\beta = (\tau_{oc}\cos\beta - \tau_{on}\sin\beta) + \tan\beta(\tau_{oc}\sin\beta +$$

$$\tau_{on}\cos\beta) = \tau_{oc}\sec\beta. \qquad \ldots(38)$$

Thus the friction term appearing in equation (23) is only related to the wall angle, ε, inasmuch as

$$\tau_{oc} = \tau_o \cos\varepsilon$$

so that if ε is not too large the influence of wall angle is not very important. In the case when blade tips are immersed in the annulus layer, the rotor (or stator) surface is moving relative to the blades. Then neglecting again the wall angle influence, it may be assumed that the shear stress is directed along the direction of the appropriate free-stream velocity vector relative to that surface.

There would seem to be no value in adapting something like the Johnson cross-flow profile in this situation. It would seem more logical that the α-parameter should be linked to the shape of the collateral velocity profile, which in turn must reflect the influence of wall shear. A proposal for a suitable profile is that the entire extent of the profile should be regarded as being influenced by secondary flow but at the wall the velocity may have a non-zero value and indeed the wall shear stress can be made to depend upon this slip velocity (or the equivalent).

Choosing, for example, a quadratic form for the collateral profile as follows:

$$\bar{V}_c/V_e = 1 - \alpha(1 - \eta')^2 \qquad \ldots(39)$$

where α is the ratio $1 - \bar{V}_c(0)\big/V_e$ and η' is y/δ, not y/h. The shear stress must be made to depend on this ratio, so that a complete solution may be obtained to the system, equations (23a) and (12a). The parameter, α, can be easily related to form factor, H; thus when $\alpha = 1$ the highest value of H occurs (2.5) while for $\alpha = 0$ or less than zero we obtain the lowest form factors. However, if a formula for skin friction of the type

$$C_f = f(H)/R_\theta^m \qquad \ldots(40)$$

is used, then $\alpha = 0$ would produce an infinite skin friction; α is thus restricted to the range 0 to 1.

The profile of equation (39) gives therefore no possibility of reverse flow at the wall. Alternative single parameter profiles could be proposed.

The main reason for choosing a profile of the type given in equation (39) is because the velocity in the laminar sub-layer rises very steeply up to values in the region of half the free-stream value; there is therefore a case for ignoring the thickness of this region in relation to the remainder of the annulus wall boundary layer.

6. Numerical Solution Procedure

It is unlikely that it will be convenient to preserve analytical functions for all of the mean and disturbance velocity components and their products for subsequent integration and differentiation with respect to the parameter, α, which is now a profile shape parameter defined by equation (39).

Referring to equations (23a) and (12a), many integrals occur in which derivatives with respect to α are involved. Thus, for example, the function $H(\alpha,\eta) = (\overline{u'^2} - \overline{v'^2})/u_e^2$ gives rise to integrals

$$\int_0^1 H(\alpha,\eta)d\eta \quad \text{and} \quad \int_0^1 H_\alpha(\alpha,\eta)\, d\eta$$

If $H(\alpha,\eta)$ is given as an array of numbers corresponding to a given α value over the range $0 \leq \eta \leq 1$ then the second integral may be found from the first from the formula

$$\int_0^1 H'(\alpha,\eta)d\eta = \frac{2}{\Delta\alpha}\left\{\int_0^1 H(\alpha + \delta\alpha,\eta)d\eta - \int_0^1 H(\alpha - \Delta\alpha,\eta)d\eta\right\} \qquad \ldots(41)$$

provided $\Delta\alpha$ is small enough so that the product, $\frac{1}{2}\Delta\alpha^2$ times the 3rd derivative is negliginly small. In some cases integrals of the type

$$\int_0^1 H(\alpha,\eta)\, U_\alpha(\alpha,\eta)d\eta$$

occur and in these cases the differentiation must be performed before integration but no numerical difficulties are foreseen in this procedure.

The solution of equations (23a) and (12a) would proceed step-by-step from given initial conditions of the variables h and α using one of the simpler predictor-corrector methods. At each iteration the values of all the integrals in those equations would be calculated for the current values of the two dependent variables.

7. Conclusion

A method is presented for the solution of the annulus wall boundary that

(i) does not depend upon any assumption regarding the flow at blade row exit

(ii) uses secondary flow theory to determine the momentum terms arising from flow distortion

(iii) uses a new equation arising from the dissipation of mean flow energy into secondary flow energy

(iv) eliminates the blade forces between the axial and tangential momentum equations

(v) from the resulting two simultaneous equations determines the values of boundary-layer thickness and velocity profile parameter.

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SYMBOLS

a,b	tangential co-ordinates of blade surfaces across the channel
B	function controlling streamwise vorticity, equation (27)
F,G,H,P,Q,U,W	integrals of type described in equation (25)
F_x	axial blade force per unit mass
F_z	tangential blade force " "
H	form factor of collateral velocity profile
h	outer limit of true boundary layer
k	disturbance kinetic energy
L	axial row length
n	co-ordinate normal to channel direction
p	pressure
q	total local velocity
s	blade pitch
S'	normal blade pitch
V	total velocity
u,v,w	cartesian velocity components
x	axial co-ordinate
y	co-ordinate normal to annulus wall
z	tangential co-ordinate
α	parameter controlling functions in equations (23a), (12a)

β/

β	flow angle (absolute) to axial direction in mainstream
$\beta^{(r)}$	flow angle (relative) to axial direction in mainstream
δ	thickness of collateral profile
δ_x^*, δ_z^*	displacement thicknesses of axial and tangential mean velocities
ε	wall angle of shear relative to mainstream
ζ	vorticity
θ_x, θ_z	momentum thicknesses of axial and tangential mean velocities
ρ	density
σ	axial solidity, L/s
τ	shear stress
τ_o	wall shear stress

Suffixes

$-$	tangential average
$'$	local disturbance from average value
1	row inlet
2	row outlet
c	collateral direction (main streamwise)
e	edge of boundary layer
s	streamwise direction
m	centre of channel
n	along co-ordinate n

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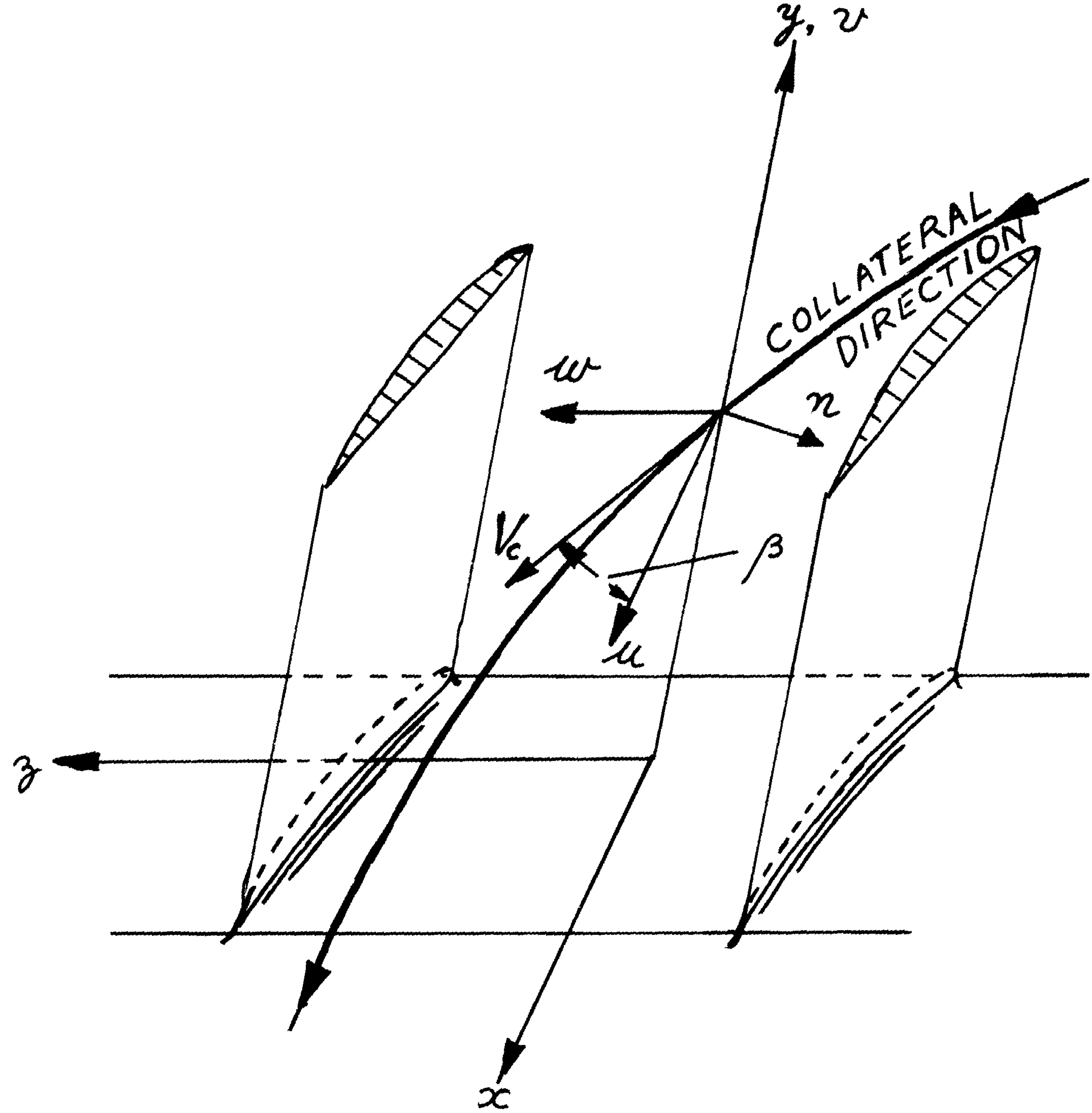

FIG. 1 CO-ORDINATE SYSTEM

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