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Some Remarks on the Induced Velocity Field of a Lifting Rotor and on Glauert's Formula

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SOME REMARKS ON THE INDUCED VELOCITY FIELD OF A

LIFTING ROTOR AND ON GLAUERT'S FORMULA

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Summary

The induced velocity field of a lifting rotor is discussed in relation to the pressure field rather than the vortex wake in an attempt to obtain a clearer understanding of the relationship between the induced velocity and rotor forces. A number of results are derived and, where appropriate, are compared with those obtained from the theory of the vortex wake.

An investigation into the validity of Glauert's formula indicates that it appears to be true for all rotor loadings for the linearized, "high speed", case.

Calculations of the induced power show that for typical rotor loadings the power in hovering flight is about 10 per cent greater than the "ideal" induced power, rising to about 15 to 20 per cent greater in forward flight.

1. INTRODUCTION

A knowledge of the induced velocity at a helicopter rotor is essential for the calculation of rotor blade forces and moments. For many applications a detailed knowledge of the complicated induced velocity field is unnecessary and Glauert's formula[9] for the mean induced velocity often gives acceptable accuracy. Glauert's formula has sometimes been misinterpreted, however, and used wrongly as a basis for detailed calculations of the induced velocity field.

This note considers the pressure field of the lifting rotor, rather than the vortex wake, in an attempt to obtain a clearer understanding of the relationship between the induced velocity and the rotor forces. Certain symmetry relations, obtained from the theory of the vortex wake, are used to investigate the validity of Glauert's formula. The discussion throughout is based on a linearized analysis.

The work of this note has probably no practical application but it is hoped that it leads to a better insight into the development of the induced velocity field.

2. EQUATIONS OF MOTION

Euler's equation of motion for a fluid is

$$\bar{a} = \frac{\partial \bar{q}}{\partial t} + \bar{q}.\nabla \bar{q} = -\frac{1}{\rho}\nabla p \qquad\qquad .(1)$$

If the motion is regarded as being that of a small disturbance $\bar{v}$ superimposed on a uniform flow of velocity $\bar{V}$, i.e. $\bar{q} = \bar{V} + \bar{v}$ such that squares and products of the components of $\bar{v}$ are negligible, Euler's equation becomes

$$\frac{\partial \bar{q}}{\partial t} + \bar{V}.\nabla \bar{v} = -\frac{1}{\rho}\nabla p \qquad\qquad . (2)$$

Taking the divergence of both sides of (2), and remembering that $\nabla.\bar{q} = \nabla\bar{v} = 0$ by continuity, we have, if ρ is constant,

$$-\frac{1}{\rho}\nabla^2 p = \nabla\left(\bar{V}.\nabla\bar{v}\right) = 0 \qquad \text{since } \bar{V} \text{ is a constant vector}$$

i.e.
$$\nabla^2 p = 0$$

Thus, for the linearized problem, the pressure field satisfies Laplace's equation, even when the flow is unsteady. For incompressible flow, (2)

can also be written

$$\frac{\partial \bar{v}}{\partial t} + \bar{V}.\nabla \bar{v} = -\nabla \left(\frac{p}{\rho}\right) = -\nabla \Phi \qquad . \qquad . \qquad . \quad (3)$$

giving

$$\nabla^2 \Phi = 0$$

Φ is known as the "acceleration potential" since, as can be seen from (3) the gradient of Φ gives the fluid acceleration.

The "acceleration potential", first introduced by Prandtl in 1936, has been used by Krienes[1] and Kinner[2] to find the loading of elliptic wings of arbitrary aspect ratio and circular wings respectively. The latter formed the basis of Mangler's work[3] in which he calculated the induced velocity distribution of a disc carrying a load similar to that of a helicopter rotor.

We illustrate the calculation of the velocity components by considering the particular case in which $\bar{V} = V\hat{\imath}$. Then, if $\bar{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ (2) becomes, for the steady case,

$$V \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \qquad . \qquad . \qquad . \qquad . \qquad . \quad (4)$$

$$V \frac{\partial v}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial y} \qquad . \qquad . \qquad . \qquad . \quad (5)$$

$$V \frac{\partial w}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial z} \qquad . \qquad . \qquad . \quad (6)$$

Assuming the disturbance velocity to be zero at a great distance ahead of the disturbing load, the velocity components anywhere in the flow field can be obtained by integration. For example, equation (6) gives

$$w = -\frac{1}{\rho V} \int_{\infty}^{x} \frac{\partial p}{\partial z} dx \qquad . \qquad . \qquad . \qquad . \quad (7)$$

The problem, briefly, is to construct solutions to Laplace's equation satisfying the appropriate boundary conditions and then to obtain the disturbance velocities from integrations such as (7). In Mangler's work Laplace's equation was found to be satisfied by a sequence of associated Legendre functions which, in addition, gave the required pressure jump across the disc. These could be regarded as a series of pressure "mode shapes" and it was found that the first two, in a suitable combination, gave an acceptable representation of a helicopter rotor loading.

3. THE UNIFORMLY LAODED DISC

The uniformly loaded disc is one which leads to comparatively simple solutions and gives a useful insight into the mechanism of the development

of the induced velocity field.

It is well-known from potential theory that a function which satisfies Laplace's equation and gives a discontinuity across a surface element is

$$d\Phi = \frac{\Delta p}{4\pi\rho}\, \hat{n}.\nabla\left(\frac{1}{r}\right) dS \qquad . \qquad . \qquad . \qquad . \qquad (8)$$

where $\hat{n}$ is the unit normal to the surface and ∇p is, in the terms of our problem, the pressure difference across the rotor disc. Equation (8) is the field due to a three dimensional doublet or dipole[4].

For a uniform pressure distribution

$$\Phi = \frac{\Delta p}{4\pi\rho}\int_S \hat{n}.\nabla\left(\frac{1}{r}\right) dS \qquad . \qquad . \qquad . \qquad (9)$$

where the integration is taken over the whole surface, which need not be necessarily plane or closed.

Now the solid angle φ which the surface subtends at any point can be expressed[4] as

$$\varphi = -\int_S \hat{n}.\nabla\left(\frac{1}{r}\right) dS$$

so that (9) can be written as

$$\Phi = \frac{-\Delta p}{4\pi\rho}\,\varphi$$

or

$$p = \Delta p\varphi/4\pi \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (10)$$

that is, the pressure at any point in the field can be expressed very simply in terms of the solid angle subtended by the surface. A sketch of the pressure field in a dimetral plane is shown in Fig.1.

Further, the acceleration field $\bar{a}$ can be obtained from (3) as

$$\bar{a} = -\nabla\Phi$$

$$= \frac{-\Delta p}{4\pi\rho}\nabla\int_S \hat{n}.\nabla\left(\frac{1}{r}\right) dS \qquad . \qquad . \qquad . \qquad (11)$$

$$= -\frac{\Delta p}{4\pi\rho}\int_S \hat{n}.\nabla\nabla\left(\frac{1}{r}\right) dS$$

By an extension of Stoke's theorem[4].

$$\int_S \hat{n}.\nabla\nabla\left(\frac{1}{r}\right) dS = -\int_c \nabla\left(\frac{1}{r}\right)\times d\bar{l}$$

where the suffix C denotes the boundary of the surface S and $d\bar{l}$ is an element of the boundary. Hence, (11) becomes

$$\bar{a} = \frac{\Delta p}{4\pi\rho} \int_C \nabla\left(\frac{1}{r}\right) \times d\bar{l}$$

$$= -\frac{\Delta p}{4\pi\rho} \int_C \frac{\bar{r} \times d\bar{l}}{r^3} \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad (12)$$

The integral on the right hand side of (12) gives the acceleration vector field due to a surface carrying a uniform pressure distribution and we note that it is identical in form to the velocity field of a vortex ring coinciding with the bounding edge C , fig.2. Katzoff and others[5] obtained this result by arguing that, since the pressure field has to satisfy Laplace's equation and the same boundary conditions as the velocity potential of a vortex ring, the two fields can be regarded as equivalent. The result was used by them to determine camber lines of wings designed to carry a uniform load.

4. CALCULATION OF THE NORMAL INDUCED VELOCITY

Since the pressure field satisfies Laplace's equation it is completely determined once the pressure distribution is prescribed. It is also independent of the disc incidence. To find the "induced velocity", i.e. the velocity components normal to the rotor disc, we note that the appropriate velocity components, found by integrating equations (4), (5), (6), will be those along and perpendicular to the flight direction. Thus, if u' and w' are the velocity components along and perpendicular to the flight direction, fig.3, the velocity normal to the disc is given by

$$w = u'\cos\chi + w'\sin\chi$$

where χ is the angle between the flight direction and the normal to the rotor disc. Then, from equations (4) and (6)

$$w = -\frac{1}{\rho V}\int_{-\infty}^{x'}\frac{\partial p}{\partial x'}\cos\chi\, dx' - \frac{1}{\rho V}\int_{-\infty}^{x'}\frac{\partial p}{\partial z'}\sin\chi\, dx' \, . \qquad \cdot \qquad \cdot \qquad (13)$$

The pressure gradient normal to the plane of the disc can be found from the tables given by Kuchemann and Weber[6]. The values along a diameter and along the axis perpendicular to the disc are shown in fig.4.

From equation (12) and fig.4 we easily find that this component is

$$\frac{\partial p}{\partial \mathfrak{z}} = \frac{\Delta p R}{4\pi} \int_0^{2\pi} \frac{r\cos(\psi - \psi') - R}{\left[\mathfrak{z}^2 + r^2 + R^2 - 2rR\cos(\psi - \psi')\right]^{3/2}} d\psi' \qquad . \quad . \quad .(14)$$

In particular, along the axis of the disc, $r = 0$ we have

$$\frac{\partial p}{\partial \mathfrak{z}} = -\tfrac{1}{2}\Delta p R^2 \left(\mathfrak{z}^2 + R^2\right)^{-3/2} \qquad . \qquad . \qquad . \quad .(15)$$

Integrating, we get

$$p - p_0 = -\tfrac{1}{2}\Delta p \left[1 + \frac{\mathfrak{z}}{(\mathfrak{z}^2 + R^2)^{1/2}}\right], \text{ for } \mathfrak{z} < 0 \qquad . \quad .(16)$$

$$p - p_0 = \tfrac{1}{2}\Delta p \left[1 - \frac{\mathfrak{z}}{(\mathfrak{z}^2 + R^2)^{1/2}}\right], \text{ for } \mathfrak{z} > 0 \qquad . \quad . \quad .(17)$$

This variation is shown in fig.5

The induced velocity for the axial (propeller) case is found by putting $\chi = 0$ in (13), giving, for any r,

$$w = -\frac{1}{\rho V}\int_\infty^{x'} \frac{\partial p}{\partial x'} dx' = -\frac{1}{\rho V}\int_\infty^{\mathfrak{z}} \frac{\partial p}{\partial \mathfrak{z}} d\mathfrak{z} \qquad . \qquad . \qquad .(18)$$

Then, integrating (15) and ensuring that $w = 0$ as $\mathfrak{z} \to \infty$ i.e. upstream of the disc, we have

$$w = \frac{-\Delta p}{2\rho V}\left\{1 + \frac{\mathfrak{z}}{(\mathfrak{z}^2 + R^2)^{1/2}}\right\} \qquad . \qquad . \qquad . \quad .(19)$$

The axial variation of induced velocity is shown in fig.6 and we see that the slipstream velocity has reached nearly 90 per cent of its ultimate value within a diameter's distance of the disc.

The integration of (4), which corresponds to the linearized form of Bernoulli's equation, shows that the axial velocity component depends only on the local pressure, i.e. $u = -\frac{p}{\rho V} +$ constant. Thus justifies, for the linear theory at any rate, the idea of the "independence of blade elements" of elementary propeller theory.

For points downstream of the disc, and for all points for which $r > R$ the constant in the above expression for u is zero and the axial velocity

component is simply proportional to minus the local pressure. Now, we saw earlier that the local pressure is proportional to the solid angle subtended by the disc, so that upstream of the disc the axial velocity must be continuous everywhere. It also follows from the sketch of the pressure field, fig.1, that the axial velocity is zero in the plane of the disc ($r > R$), and that it actually becomes negative as we proceed downstream. However, since we know that the velocity downstream of the disc continually increases, there must be a discontinuity of velocity, i.e. a definite wake, at $r = R$ for points behind the disc, as is assumed, of course, in the vortex theory of propellers.

Koning[7] has also investigated the radial component of velocity for a uniformly loaded disc, but the impression given in the work that a discontinuity extends ahead of the disc is false and is a consequence of the assumptions made to simplify the analysis.

5. FORWARD FLIGHT, $\chi = 90°$

Another case of interest is the "high speed" case, $\chi = 90°$. From (18) the induced velocity is

$$w = -\frac{1}{\rho V} \int_0^x \frac{\partial p}{\partial j}\, dx$$

since $x = x'$ and $j = j'$

In the method used by Katzoff and others[5] to calculate the camber of wings, the integration was first performed on an element of the contour bounding the lifting surface (equation 12). Thus, with dx', dy' denoting the components of the element dl' at point x', y' and x, y the point in the field, integration gave

$$dw = -\frac{\Delta p}{4\pi\rho V}\left\{\frac{(x-x')dx' + (y-y')dy'}{(y-y')\left[(x-x')^2 + (y-y')^2\right]^{1/2}} + \frac{dx}{y-y'}\right\} \qquad .(20)$$

Expressing these quantities in terms of polar rotor coordinates and integrating over the circular contour of the rotor disc, we find after a little manipulation that

$$w = -\frac{\Delta p}{4\pi\rho V}\int_0^{2\pi}\frac{\mu \sin(\psi-\psi')\,d\psi'}{(\mu\sin\psi - \sin\psi')\left[1+\mu^2 - 2\mu\cos(\psi-\psi')\right]^{1/2}} + \frac{\Delta p}{4\pi\rho V}\int_0^{2\pi}\frac{\sin\psi'\,d\psi'}{\mu\sin\psi - \sin\psi'}$$

$$(21)$$

where $\mu = r/R$

Along the longitudinal axis of symmetry, and ahead of the centre of the rotor, $\psi = \pi$, and (21) becomes

$$\omega = -\frac{\Delta p}{4\pi\rho V} \left\{ 2\pi - \int_0^{2\pi} \frac{\mu\, d\psi'}{\sqrt{1+\mu^2 - 2\mu\cos\psi'}} \right\}$$

$$= \frac{-\Delta p}{4\pi\rho V} \left\{ 2\pi - 4\mu K(\mu) \right\} \quad \text{for } \mu < 1 \qquad . \qquad . \qquad (22)$$

$$\left[\text{For } \mu > 1 \quad \mu K(\mu) = K\left(\tfrac{1}{\mu}\right) \right]$$

where $K(\mu)$ is the complete integral of the first kind.

Similarly, along the rear longitudinal axis $\psi = 0$, we have

$$\omega = -\frac{\Delta p}{4\pi\rho V} \left\{ 2\pi + 4\mu K(\mu) \right\} , \quad \text{for } \mu < 1 \qquad . \qquad . \qquad (23)$$

The induced velocity along the longitudinal axis is shown in fig.7.

Along the lateral axis of the rotor, $\psi = \pi/2$,we have

$$\omega = -\frac{\Delta p}{4\pi\rho V} \left\{ 2\pi - \int_0^{2\pi} \frac{\mu\, d\psi'}{\mu - \cos\psi'} \right\} \qquad . \qquad . \qquad (24)$$

Now

$$\int_0^{2\pi} \frac{\mu\, d\psi'}{\mu - \cos\psi'} = 1 , \quad \text{for } \mu < 1$$

$$= \frac{\mu}{\sqrt{\mu^2-1}} , \quad \text{for } \mu > 1$$

Hence ω has the constant value $-\Delta p/2\rho V$ along the lateral axis within the disc, and $-\frac{\Delta p}{2\rho V}\left\{1 - \frac{\mu}{\sqrt{\mu^2-1}}\right\}$ outside the disc.

6. MOMENTUM CONSIDERATIONS

At this stage it is useful to consider the relationship between the pressure jump at a position on the disc in forward flight and the induced velocity there. In axial flight, assuming the rotation of the slipstream to be negligible, we have the well-known relationship between the thrust and the induced velocity v_i in the form

$$T = 2\rho A \left(V + v_i\right) v_i \qquad . \qquad . \qquad . \qquad (25)$$

and, for an arbitrary loading, the differential form

$$dT = 2\rho \left(V + v_i\right) v_i\, dA \qquad . \qquad . \qquad . \qquad (26)$$

is assumed to hold. The validity of this relation, as stated by Glauert[7], has never been established and in a recent paper[8] it is shown to be untrue in general. It is true for the linear case as has been shown earlier.

The simplicity of this result arises from the fact that the induced velocity increment is in the same direction as the general flow, and that the flow behind the disc is confined to a well-defined slipstream. For these reasons it is easy to apply momentum principles and arrive at (26).

Now, in order to obtain a mean induced velocity for calculating rotor performance, Glauert[9], in 1926, proposed the formula

$$T = 2\rho A V' v_i' \qquad . \qquad . \qquad . \qquad . \qquad (27)$$

where
$$V' = \sqrt{V^2 + v_i^2}$$

Although no proof of (27) has been given, the justification claimed for this formula is that it reduces to (25) in the hovering case, $V = 0$ and assumes the correct form for the induced velocity of an elliptically loaded wing when V is large.

An interpretation of the formula (28) is to imagine a cylindrical slip-stream, having the same diameter as the rotor, impinging upon the rotor at speed V and being deflected downwards by it so that it ultimately acquires a downwash component of $2v_i$. By applying momentum principles (27) is readily obtained. The only virtue of such a fictitious representation is that it leads easily to (27) but the impression has often been gained that it is "obvious" from momentum considerations and, indeed, that a differential form,

$$dT = 2\rho V' v_i' \, dA \qquad . \qquad . \qquad . \qquad . \qquad (28)$$

analogous to (26), is also applicable.

Now, although the lift of the rotor must be accountable in terms of rate of change of momentum of the air, momentum considerations are by no means simple when the rotor is inclined to the flight path and the induced velocity can no longer be expressed simply in terms of the local loading, as in (26). To see this we refer back to the case, $\chi = 90°$ which we solved analytically, (22) and (23),and consider a physical interpretation of this integration. The function being integrated is the pressure gradient $\partial p / \partial z$ shown in fig.4, which we have seen to be proportional to the local vertical acceleration. As the air approaches the rotor it comes under the influence of the pressure field which imparts an upwards velocity, i.e. an upwash, in front of the disc. As the air moves downstream of the rotor's leading edge the pressure gradient reverses and a downwash develops some distance behind the leading edge. At the trailing edge the pressure gradient again reverses and the downwash gradually decreases finally becoming twice

the value at the centre of the rotor. In this special case the slipstream boundary, or vortex wake, degenerates into a flat strip; there is no "flow" through it, the momentum changes which occur being associated with changes of the direction of the air in the neighbourhood of the rotor, that is, in the conversion of an upwash into a downwash. In the case of inclined flow it can be seen physically, and also from (13), that the induced velocity at a particular point of the rotor depends on the variation of the pressure gradient along the path of integration and not simply on the local discontinuity, as happens to be true in the axial case. Thus (28) cannot be valid in general and attempts to use such a relationship to connect the local rotor loading with the induced velocity are fallacious. Such attempts, in any case, usually require a special imterpretation of the increment of mass flow $\rho V' dA$.

Another point worth noting is that, although there is a pressure discontinuity at the disc, the gradient, and hence, the flow acceleration, is continuous, as is also the flow velocity. There is no sudden deflection of the flow field as is sometimes thought to be the case in supposing (28) to be true.

Further, since any axially symmetrical loading can be build up of an assembly of elementary concentric circular loadings, the centre of the rotor occupies the same geometric position in every such elementary loading. Hence, the induced velocity at the rotor centre depends only on the total local loading; if it is zero, as would be expected, the induced velocity must be zero there also. This does not hold for any other point on the rotor, i.e. the local induced velocity, generally, will depend on the shape of the loading distribution as well as the magnitude.

7. GENERAL FORWARD FLIGHT CASE

In the general case the flow will be inclined at an angle χ to the rotor disc and we have the relationships

$$x' = x \sin\chi + \zeta \cos\chi$$

$$\zeta' = -x \cos\chi + \zeta \sin\chi$$

(13) then becomes

$$w = -\frac{1}{\rho V} \int_{\infty}^{x} \frac{\partial p}{\partial \zeta} \sin\chi \, dx - \frac{1}{\rho V} \int_{-\infty}^{\zeta} \frac{\partial p}{\partial \zeta} \cos\chi \, d\zeta$$

But along the path of integration, i.e. along a path parallel to the x'-axis $dx = -d\zeta \tan\chi$ and so

$$w = -\frac{1}{\rho V} \int_{-\infty}^{\zeta} \frac{\partial p}{\partial \zeta} \sec\chi \, d\zeta$$ (29)

where, in the integrand, x and z are related by the path of integration. Now (14) can be written

$$\frac{\partial p}{\partial z} = \frac{\Delta p R}{4\pi} \int_0^{2\pi} \frac{(x\cos\psi' + y\sin\psi' - R)d\psi'}{\left[x^2 + y^2 + z^2 + R^2 - 2xR\cos\psi' - 2yR\sin\psi'\right]^{3/2}} \cdot \qquad (30)$$

and along the path of integration

$$x = r\cos\psi - z\tan x = r\cos\psi - mz$$

where $m = \tan x$,

and $y = r\sin\psi = \text{constant}.$

Hence for a point on the disc

$$\omega = \frac{\Delta p R\sqrt{1+m^2}}{4\pi\rho V} \int_0^\infty \int_0^{2\pi} \frac{\{R - mz\cos\psi' + r\cos(\psi - \psi')d\psi'dz}{\left[z^2(1+m^2) + r^2 + R^2 - 2mrz\cos\psi - 2mRz\cos\psi' + 2rR\cos(\psi-\psi')\right]^{3/2}} \qquad (31)$$

Apart from a slight difference of notation and definition of the angle corresponding to ψ' the above integral is identical to that of ref.10 in which the induced velocity of a uniformly loaded rotor had been calculated by integrating the effect of an inclined cylindrical vortex wake. Another difference is that the induced velocity given here is expressed directly in terms of the pressure jump instead of the circulation of a unit slice of the vortex wake. Unfortunately, the integral cannot be evaluated analytically for a general point on the rotor disc except for points on the longitudinal axis of symmetry, but even then, as shown in ref.10, the result can only be expressed of elliptic integrals of the first and third kinds. The special case, $x = 90°$, agrees with the results given in (22) and (23).

8. THE SYMMETRY RELATIONS

A circular disc carrying a uniform load generates an elliptical vortex wake. By superimposing a skew-symmetric wake on the original one, a two-dimensional elliptic wake is created by means of which Katzoff[11] obtained certain relationships between the induced velocity components. They are

(i) If P and Q are two points on the rotor disc symmetrically located about the lateral axis, the sum of the induced velocity components ω_p and ω_Q is equal to the vertical component of the induced velocity field within the two-dimensional wake. Since this is constant, it follows that

$\omega_p + \omega_Q$ is also constant and the induced velocity distribution is skew-symmetrical with respect to the lateral axis.

(ii) If P and Q are symmetrically located about the lateral axis of the disc and lie outside it we have

$$\omega_p + \omega_Q = v'\sin x . \qquad (32)$$

where v' is the longitudinal* component of velocity in the ellipse plane at the point corresponding to P (or Q), fig.8. Now, the flow about the elliptical wake is due to the motion it induces upon itself relative to the surrounding air. The component of this motion which gives rise to the flow about the elliptical wake is the velocity normal to the wake axis, U say. By considering the circulation round a circuit threaded into and out of the wake, it has been shown in ref.10 that U is $\tan \chi/2$ times the velocity within, and parallel to, the ultimate wake and that this latter component is twice the value v_i at the rotor centre; that is

$$U = 2v_i \tan \chi/2 \qquad . \qquad . \qquad . \qquad (33)$$

Ref.11 gives the complex potential of the ellipse flow as

$$\varphi + i\psi = -iUa\sqrt{\frac{a+b}{a-b}}\, e^{-\zeta} \qquad . \qquad . \qquad . \qquad (34)$$

where $\zeta = \xi + i\eta$ are elliptical coordinates related to the cartesian coordinates by $x' + iy' = z' = c\cosh\zeta$, a and b being the semi-major axes of the ellipse and $c^2 = a^2 - b^2$. In our case $a = R$, $b = R\cos\chi$ giving $c = R\sin\chi$ so that, with (33), equation (34) in terms of z' becomes

$$\varphi + i\psi = -2iv_i\,\text{cosec}\,\chi \left[z' - \sqrt{z'^2 - R^2\sin^2\chi} \right] \qquad . \qquad . \qquad . \qquad (35)$$

Differentiating (35) with respect to z' gives the velocity components as

$$u - iv' = 2iv_i\cos\text{ec}\,\chi \left[1 - \frac{z'}{\sqrt{z'^2 - R^2\sin^2\chi}} \right]$$

Now, on the lateral axis $\omega_P = \omega_Q = \omega$ say and $z' = x'$. Then, from (32) and writing $x' = \mu R$ and $\sin\chi = \gamma$, we finally have

$$\frac{\omega}{v_i} = 1 - \frac{\mu}{\sqrt{\mu^2 - \gamma^2}} \qquad . \qquad . \qquad . \qquad (36)$$

Equation (36), shown plotted in fig.(9), gives the distribution of induced velocity along the lateral axis as a fraction of the value at the rotor centre. These values have also been given in ref.12, together with

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* Here, as in ref.11, we adopt the conventional axes system for the ellipse, i.e. the x' axis corresponds to the major axis and the y' axis to the minor axis. Thus, we have $x' = \gamma$ and $y' = x\cos\chi$.

other values in the lateral plane, but the calculations were obtained by a numerical evaluation of an integral derived from (31) with $\psi = \frac{\pi}{2}$. However, for the special case of the lateral axis, it has been found possible to rearrange the integral and solve it exactly, giving the same result as (36).

9. VERIFICATION OF GLAUERT'S FORMULA

Let x, y be coordinates in the rotor plane and x', y' coordinates in the plane perpendicular to wake axis, fig.(10). The cross-section of the wake parallel to the rotor plane is, of course, a circle with radius R

Consider now an annulus concentric with the rotor and lying outside it, fig.(11a). The mean induced velocity $\bar{\omega}$ over the annulus is

$$\bar{\omega} = \frac{1}{2A} \iint \left(\omega_p + \omega_Q \right) dx\,dy \qquad . \qquad . \qquad . \qquad (37)$$

where as in the previous section, ω_p and ω_Q are the induced velocities at points equally spaced about the lateral axis and A is the area of the annulus. From (32) equation (37) can be written as

$$\bar{\omega} = \frac{\sin \chi}{A} \iint v'\,dx\,dy \qquad . \qquad . \qquad . \qquad (38)$$

If φ and ψ are the velocity and stream functions of the flow about elliptical wake,

$$v' = \frac{\partial \varphi}{\partial y}' = -\frac{\partial \psi}{\partial x}',$$

The coordinates in the rotor and ellipse plane are related by

$$x = x' , \qquad y = y' \sec \chi$$

so that

$$v' \left(1 + \cos \chi \right) = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x}$$

and $\bar{\omega}$ can be written as

$$\bar{\omega} = \frac{\sin \chi}{A(1 + \cos \chi)} \iint \left(\frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x} \right) dx\,dy \qquad . \qquad . \qquad (39)$$

But, by Green's theorem,

$$\int_c \left(\varphi - i\psi \right)\left(dx + i\,dy \right) = \int_c \left(\varphi\,dx + \psi'\,dy \right) - i \int \left(\psi\,dx - \varphi\,dy \right)$$

$$= \iint \left(\frac{\partial \psi}{\partial x} - \frac{\partial \varphi}{\partial y} \right) dx\,dy + i \iint \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} \right) dx\,dy$$

so that, from (39),

$$\bar{w} = -\frac{\sin\chi}{A(1+\cos\chi)} \, Rl \int_c (\varphi - i\psi)\, dz$$

where the suffix C denotes the contour of the annulus, fig.(11a).

Now, from the previous section, we have

$$\varphi - i\psi = 2iv_i\cosec\chi \left[\bar{z}' - \sqrt{\bar{z}'^2 - R^2\sin^2\chi} \right]$$

giving

$$\bar{w} = \frac{2v_i}{A(1+\cos\chi)} \, Rl \int_c i\left[\bar{z}' - \sqrt{\bar{z}'^2 - R^2\sin^2\chi} \right] dz . \quad (40)$$

The contour consists of two circles of radius R and μR say, where $\mu > 1$. Although the function in the integrand has branch points, they both lie within the inner circle so that no discontinuity of the function occurs between the straight portions AD and BC . Hence, the integrals along these portions cancel and we need only consider integration round the circular arcs $r = R$ and $r = \mu R$. If $z = re^{i\psi}$ we have

$$\int_c (\varphi - i\psi)\, dz = i\int_0^{2\pi} (\varphi_1 - i\psi_1) Re^{i\psi}d\psi - i\mu\int_0^{2\pi} (\varphi_2 - i\psi_2) Re^{i\psi}\, d\psi$$

where the suffixes 1 and 2 denote the values on the inner and outer circles respectively. On the inner circle $x' = R\cos\psi$ and $y' = R\sin\psi\cos\chi$ and, therefore,

$$\varphi_1 - i\psi_1 = 2iRv_i\cosec\chi \left[\cos\psi - i\sin\psi\cos\chi - \sqrt{(\cos\psi - i\sin\psi\cos\chi)^2 - \sin^2\chi} \right]$$

$$= 2iRv_i\cosec\chi \left[\cos\psi - i\sin\psi\cos\chi - (\cos\psi\cos\chi - i\sin\psi) \right]$$

$$= 2iRv_i \tan\tfrac{\chi}{2}\, e^{i\psi}$$

and, hence,

$$\int_0^{2\pi} (\varphi - i\psi)\, dz = -2R^2 v_i \tan\tfrac{\chi}{2} \int_0^{2\pi} e^{2i\psi}\, d\psi$$

$$= 0$$

On the circle of radius μR

$$\varphi_2 - i\psi_2 = 2iRv_i \cosec\chi \left[\mu(\cos\psi - i\sin\psi\cos\chi) - \sqrt{\mu^2(\cos\psi - i\sin\psi\cos\chi)^2 - \sin^2\chi} \right]$$

Considering the first term, the contribution to the total integral is

$$2R^2 v_i \cosec\chi \int_0^{2\pi} \mu^2 (\cos\psi - i\sin\psi\cos\chi)(\cos\psi + i\sin\psi) \, d\psi$$

of which the real part is easily found to be

$$2\pi\mu^2 R^2 v_i \cosec\chi(1 + \cos\chi) = 2\pi\mu^2 R^2 v_i \cot\chi/2$$

The second integral can be evaluated by reverting back (in terms of unit radius) to cartesian coordinates in complex form by $\zeta = e^{i\psi}$ so that

$$\cos\psi = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right), \quad i\sin\psi = \frac{1}{2}\left(\zeta - \frac{1}{\zeta}\right) \quad \text{and} \quad e^{i\psi} d\psi = -i \, d\zeta$$

Then, if

$$I = \int_0^{2\pi} \sqrt{\mu^2(\cos\psi - i\sin\psi\cos\chi)^2 - \sin^2\chi} \; e^{i\psi} d\psi$$

we have

$$I = -i \int_c \sqrt{\frac{\mu^2}{4}\left\{\left(\zeta + \frac{1}{\zeta}\right) - \left(\zeta - \frac{1}{\zeta}\right)\cos\chi\right\}^2 - \sin^2\chi} \; d\zeta$$

where the contour is the unit circle about the origin. On rearranging we find

$$I = -\frac{i\mu}{2}(1 + \cos\chi)\int_c \sqrt{\left(1 + \zeta\tan^2\chi/2\right)^2 - \left(\frac{2\zeta\tan\chi/2}{\mu}\right)^2}$$

The only pole is a simple pole at $\zeta = 0$. Expanding the term under the root sign we easily find the residue of the integrated to be unity. Hence

$$I = \pi\mu(1 + \cos\chi)$$

and the contribution due to the term in I is found to be $2\pi\mu^2 R^2 v_i \cot\chi/2$ This term cancels with the first one so that, on the outer circle, the real part of $\int (\varphi - i\psi) \, d\zeta$ is zero; in other words, the mean value of the induced velocity over any annulus concentric with a circular disc carrying a uniform load is zero.

The implication of this result is, as follows:- We saw in section 8 that the induced velocity distribution on the rotor is skew-symmetric with

respect to the lateral axis. Hence, the mean value of the induced velocity over the rotor must be the same as the (constant) value v_ι' on the lateral axis. To find this value we put $r = 0$ in (31) and, by means of standard tables, we perform the integration with respect to ψ. This is easily found to give

$$v_\iota' = \frac{\Delta p}{4\pi\rho V} \int_0^{2\pi} d\psi$$

$$= \frac{\Delta p}{2\rho V}$$

But Δp is the thrust loading T/A so

$$v_\iota' = \frac{T}{2\rho A V} \qquad\qquad\qquad\qquad \cdot (41)$$

This is Glauert's formula for v_ι' when V is large; that is, Glauert's formula is true for a uniformly loaded disc. This result has also been obtained by Shaidakov[13] who related the thrust and induced velocity to the impulse and circulation of the vortex rings shed by the disc.

Now suppose we have a uniformly loaded rotor of radius R carrying a thrust T_1, say. Then, by (41) the mean induced velocity is

$$v_{\iota_1} = \frac{T_1}{2\rho\pi R^2 V}$$

If a concentric uniform loading is added over a circular disc of radius R_2 $\left(R_2 < R\right)$ the mean induced velocity v_{ι_2} over the disc is

$$v_{\iota_2} = \frac{T_2}{2\rho\pi R_2^2 V}$$

But the mean induced velocity over the annulus between $r = R_2$ and $r = R$ has been found to be zero; therefore, over the whole rotor,

$$\int v_\iota \, dA = v_{\iota_1} \pi R^2 + v_{\iota_2} \pi R^2$$

$$= \frac{T_1 + T_2}{2\rho V}$$

i.e. the mean induced velocity $\bar{v}_\iota$ for the combined loading, is given by

$$\bar{v}_\iota = \frac{T_1 + T_2}{2\rho\pi R^2 V} = \frac{T}{2\rho\pi R^2 V}$$

where T is the total thrust.

Thus, for two concentric circular loadings the mean induced velocity, taken over the larger radius, depends only on the total thrust and not on the radius of the smaller loading. And since any axially symmetrical loading can be built up from an arrangement of such concentric loadings, Glauert's formula holds for axially symmetric loadings also.

Another case which can be examined fairly easily is the one for which the area outside the load is a circle which just touches the circular load, fig.(11b). Taking the origin as the point of contact of the circles, let ρR be the distance of the centre of the larger circle from the origin and the distance of a point on this circle. Then, as is well known from coordinate geometry, r is given by

$$ r = 2\rho R \cos(\theta - \alpha) \qquad \qquad (42) $$

where θ and α are defined in fig.(11b).

As before, let $r = \mu R$, so that (42) can be written

$$ \mu = 2\rho \cos(\theta - \alpha) \qquad \qquad (43) $$

Ignoring the constant in front of the integral (40), we wish to calculate

$$ J = Im \int \left[\bar{z}' - \sqrt{\bar{z}'^2 - R^2 \sin^2 \chi} \right] d\bar{z} $$

where the integration is taken round the contour made up of the rotor circumference, as before, and the circle (43). On the outer circle μ is no longer constant but a function of the angular coordinate θ of the point on the outer circle.

The value of J taken round the rotor circumference has already been found to be zero.

Then, if

$$ z = r e^{i\theta} = \mu R e^{i\theta}, \quad dz = R e^{i\theta} \left(i\mu + \mu \frac{d\mu}{d\theta} \right) d\theta $$

and

$$ \bar{z}' = \mu R \left(\cos\theta - i \sin\theta \cos\chi \right) $$

Also,

$$ \frac{d\mu}{d\theta} = -2\rho \sin(\theta - \alpha) \quad \text{and} \quad \mu \frac{d\mu}{d\theta} = -2\rho^2 \sin 2(\theta - \alpha) $$

With these values we easily find that $Im \int \bar{z}' dz = 2\pi \rho^2 (1 + \cos \chi)$

and, by the same means as for the previous case, we find that

$$ Im \int \sqrt{\bar{z}'^2 - R^2 \sin^2 \chi} = 2\pi \rho^2 (1 + \cos \chi) \qquad \text{,also.} $$

Hence, since these quantities cancel, the mean induced velocity taken over the lune (11b) is also zero and Glauert's formula holds for any rotor load distribution made up of such basic loadings.

For the more general case, represented by fig.(11c), it was found that the algebra became prohibitively complicated and an analytical solution was not attempted. But,

$$\bar{w} = \frac{\sin \chi}{A} \iint v' \, dx \, dy$$

$$= \frac{\sin \chi}{A} \iint \frac{\partial \varphi}{\partial y} \, dx \, dy \quad . \quad . \quad . \quad . \quad (38)$$

$$= - \frac{\sin \chi}{A} \int \varphi \, dy \quad . \quad . \quad . \quad (39)$$

by Green's theorem,

Equation (34) gives

$$\varphi = - U a \sqrt{\frac{a+b}{a-b}} \, e^{-\xi} \sin \eta \quad . \quad . \quad . \quad (40)$$

and, in terms of elliptic coordinates,

$$y = C \sinh \xi \sin \eta \quad . \quad . \quad . \quad . \quad (41)$$

Using (40) and (41), the numerical computation of (39) was found to be quite simple for the regions 11a, b, c and confirmed the results obtained analytically for regions 11a and 11b. The computation also showed that, within the numerical accuracy of the calculations, the mean value of the induced velocity over the region 11c was also zero.

Thus, we have the rather remarkable result that if a circle is drawn round a circular area carrying a uniform load, the mean induced velocity $\bar{w}$ taken over the region between these two circles is zero.

It can easily be verified that $\bar{w}$ does not vanish for every region

taken outside the circular load. For example, if we take an ellipse which
is confocal with the elliptical wake boundary and project it onto the rotor
plane, we obtain another ellipse which can be regarded as the outer contour.
Since this ellipse, and the boundary of the rotor itself, correspond to two
constant values of ξ in the elliptical coordinates, the integration of
for this case, can be performed quite easily and is found to be non-zero.

It would appear, then, that the result for the general circle, fig.(11c),
means that the mean induced velocity over the rotor is independent of the
loading since any loading can be constructed from an arrangement of circular
loadings of varying radii.

Thus, Glauert's formula for the linearized "high speed" case appears
to be true whatever the rotor load distribution. Whether it is still true
for low speeds, when the forward speed is of the same order as the induced
velocity, cannot be checked by linear theory.

10. THE INDUCED POWER IN FORWARD FLIGHT

The fact that the slipstream of a uniformly loaded rotor in hovering
and vertical flight is a distinct cylinder of moving air enables the kinetic
energy of the slipstream to be calculated easily. This leads to the well
known result that the induced power is simply the thrust multiplied by the
induced velocity. For an axially symmetric rotor loading the symmetry of
the slipstream in vertical flight enables the induced power to be calculated
quite easily by evaluating $\int v_i \, dT$ over the whole rotor. It can be shown
by means of the calculus of variations that the induced power for a given
thrust is least when the induced velocity distribution, and therefore the
rotor loading, is uniform. This is called the "ideal" induced power. It
can easily be shown that when the radial induced velocity distribution is
triangular the induced power is about 13% higher than the "ideal" value. The
induced power in vertical and forward flight is often estimated by calculating
the "ideal" power and increasing it by about 15% to take the non-uniformities
of loading and induced velocity into account.

The question is; bearing in mind that the induced velocity distri-
butions for a given loading are very different in vertical and forward flight,
is a factor of about 1.15 valid for all flight cases? To answer this we
consider a uniformly loaded rotor in forward flight and calculate the rate
at which kinetic energy is being supplied to the slipstream. In the ulti-
mate slipstream the energy consists of two contributions

    (i) the energy E_1 of the uniform flow within a unit length of the
inclined vortex wake.

    (ii) the energy E_2 of the air outside the wake due to the movement
of the wake through it.

In unit time the length of the ultimate wake increases by V units so that the power expended is $V(E_1 + E_2)$.

Now the cross-sectional area of the wake is $\pi R^2 \cos \chi$ and therefore the mass of air in a unit length is $\rho \pi R^2 \cos \chi$. Also if

v_i is the value of the induced velocity on the lateral axis of the rotor the velocity within and relative to the wake is $2 v_i$. But, as discussed in section 8, the wake is itself moving with velocity $2 v_i \tan \chi/2$ normal to its axis so that the absoulte velocity of the air in the wake is

$2 v_i \sec \chi/2$. The energy in the wake is therefore

$$E_1 = 2 \rho v_i^2 \pi R^2 \cos \chi \sec^2 \chi/2$$

To calculate the kinetic energy of the flow outside the elliptic wake we use the formula[14]

$$E_2 = -\tfrac{1}{2} \rho \int \varphi \, d\psi$$

where φ and ψ are the potential and stream functions of equation 34 evaluated on the wake boundary. On the boundary ξ has the constant value ξ_o given by $e^{-\xi_o} = \tan \chi/2$, hence

$$\varphi = -2 v_i R \tan \chi/2 \sin \eta$$

$$\psi = 2 v_i R \tan \chi/2 \cos \eta$$

giving

$$E_2 = 2 \rho v_i^2 R^2 \tan^2 \chi/2 \int_o^{2\pi} \sin^2 \eta \, d\eta$$

$$= 2 \rho \pi R^2 v_i^2 \tan^2 \chi/2$$

The induced power is therefore

$$P_i = 2 \rho \pi R^2 v_i^2 V \left(\cos \chi \sec^2 \chi/2 + \tan^2 \chi/2 \right)$$

$$= 2 \rho \pi R^2 v_i^2 V$$

$$= T v_i$$

where v_i is the induced velocity on the lateral axis.
Thus for a uniformly loaded rotor the formula for induced power is the same as in hovering and vertical flight.

It would be useful to be able to calculate the induced power for practical rotor loadings. Unfortunately the induced velocity distributions corresponding to arbitrary loadings cannot be found in general but Mangler[3] has calculated the induced velocity field for an axially symmetric loading whose radial distribution closely resembles those occuring in practice. If

$\chi = r/R$ is the non-dimensional radial coordinate of the disc Mangler's loading can be expressed as

$$\Delta p = \frac{T}{\pi R^2} \cdot \frac{15}{4} \chi^2 \sqrt{1 - \chi^2}$$

where T is the total thrust. The shape of this loading distribution is shown in fig.12. It should be noted that equation (42) implies that the blade loading is proportional to $\chi^3 \sqrt{1-\chi^2}$.

The induced velocity at the rotor corresponding to (42) can be expressed in the form

$$v_i = 4 v_{i_0} \left\{ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos n\psi \right\}$$

where v_{i_0} is the mean "momentum" induced velocity. The coefficients a_n are given by

$$a_0 = \frac{15}{8} \chi^2 \sqrt{1 - \chi^2}$$

$$a_1 = -\frac{15\pi}{256} \left(5 - 9\mu^2\right)\left(1 - \mu^2\right)^{1/2} \nu^{1/2}$$

$$a_3 = \frac{45\pi}{256} \left(1 - \mu^2\right)^{3/2} \nu^{3/2}$$

where $\mu^2 = 1 - \chi^2$ and $\nu = \frac{1 - \cos\chi}{1 + \cos\chi}$

For even values of $n \geq 2$

$$a_n = (-1)^{(n-2)/2} \frac{15}{8} \left[\frac{n+\mu}{n^2-1} \cdot \frac{9\mu^2 + n^2 - 6}{n^2 - 9} + \frac{3\mu}{n^2 - 9}\right] \left(\frac{1-\mu}{1+\mu}\right)^{n/2} \nu^{n/2}$$

and for odd values of $n \geq 5$, $a_n = 0$

Mangler also gives the induced velocity field far downstream of the rotor but the expressions are much too complicated for calculating the induced power in the same way for the uniform loading considered above. However the special case of "high speed" flight, χ = 90deg, can be calculated quite easily by making use of the formula[15] which expresses the induced drag in terms of the spanwise circulation and the normal velocity component in the far wake i.e. in the so-called Trefftz plane. If D_i is the induced drag the formula is

$$D_i = \frac{1}{2}\rho \int_{-R}^{+R} \Gamma \omega \, dr$$

giving the induced power $P_i = \frac{1}{2}\rho V \int_{-R}^{+R} \Gamma \omega \, dr$

where Γ is the local circulation and w the induced velocity. In the case of the uniformly loaded rotor the "circulation" is simply proportional to the "chord" and the distribution is therefore elliptic. It can easily be verified that the circulation corresponding to Mangler's loading is

$$\Gamma = \frac{T}{\rho V R} \cdot \frac{15}{32} \left(1 - x^2\right)\left(1 + 3x^2\right)$$

Although Mangler gave a formula for the induced velocity in the far wake the integral (45) becomes too complicated to be evaluated analytically. However, the table of values given in his original paper[16], together with equation (46), enables the integral to be evaluated numerically quite easily. The induced power can be expressed as

$$P_i = T v_{i_0} I$$

where

$$I = \frac{15}{8} \int_0^1 \bar{\omega} \left(1 - x^2\right)\left(1 + 3x^2\right) dx$$

and $\bar{\omega} = \frac{4w}{v_{i_0}}$ are the values given by Mangler. Numerical integration of I gives

$$P_i = 1.174 \, T v_{i_0}$$

$$= 1.174 \, P_{i_0}$$

where $P_{i_0} = T v_{i_0}$ is the induced power of a uniformly loaded rotor.

One should also be able to calculate the induced power by considering the backward tilt of the local blade thrust vector due to the induced velocity, as in classical blade element theory. The thrust carried on an annulus of width dr is

$$dT = 2\pi r \Delta p \, dr$$

and this thrust, of course, is shared by the b blades of the rotor. The elementary induced torque is therefore

$$dQ_i = r \, dT \frac{v_i}{\Omega r} = \frac{v_i}{\Omega} dT$$

where v_i is the local induced velocity at the blade.

The induced power contribution is

$$dP_i = \Omega \, dQ_i = v_i \, dT$$

$$= 2\pi r \Delta p \, v_i \, dr$$

Since Δp is a function of x only and v_i is periodic with respect to azimuth the <u>mean</u> value of P_i depends only on the first term of (43). Hence from (42) and (43) we find that

$$P_i = \frac{225}{8} T v_{i_0} \int_0^1 x^5 \left(1 - x^2\right) dx$$

$$= \frac{75}{64} T v_{i_0} \qquad = 1.172 T v_{i_0}$$

which, apart from a negligible numerical difference, is the same answer as before.

Now for axial flight at velocity V the momentum theory gives

$$dT = 2\pi r \Delta p \, dr = 2\rho V v_i \cdot 2\pi r \, dr$$

$$\text{or} \quad v_i = \frac{\Delta p}{2\rho V}$$

Using Mangler's loading, equation (42) gives

$$v_i = \frac{T}{2\rho V \pi R^2} \cdot \frac{15}{4} \chi^2 \sqrt{1-\chi^2}$$

$$= \frac{15}{4} v_{i_0} \chi^2 \sqrt{1-\chi^2}$$

and this is identical to the first term of the series (43). Since similar results are true of the other (elliptical) loading considered by Mangler and for the uniformly loaded rotor, it is reasonable to assume that any axially symmetric loading leads to a series of the form (43) of which the first term gives the induced velocity for the same loading in axial flight. Hence, the induced power of a symmetrically loaded rotor in forward flight is the same as that in axial flight fast enough for linearization to be valid. For the non-linear relationship of hovering flight it can easily be shown that the induced power for Mangler's loading is about 1.11 times that of a uniform load. Thus the induced power factor for Mangler's load rises from 1.11 in hovering flight to 1.17 in forward flight.

If the radial loading of the disc is of the form

$$\Delta p = C \chi^\eta$$

the induced power factor $1+k$ can be shown to be given by

$$1 + k = \frac{\left(1 + \frac{\eta}{2}\right)^{3/2}}{1 + \frac{3}{4}\eta}$$

for hovering flight, and

$$1 + k = \frac{\left(1 + \frac{\eta}{2}\right)^2}{1+\eta}$$

for forward flight.

Numerical values of these factors are given in fig.12. It should be remarked again that if the disc loading is proportional to χ^η, the corresponding blade loading is proportional to $\chi^{\eta+1}$.

For a given loading it appears, therefore, that the induced power factor in forward flight is somewhat higher than in hovering flight.

SYMBOLS

$\bar{a}$	fluid acceleration vector
p	pressure
P	induced power
P_i	ideal induced power
$\bar{q}$	fluid velocity vector
R	rotor radius
u, v, w	disturbance velocity components in rotor plane
$\bar{v}$	disturbance velocity vector
$w,$	induced velocity in plane of rotor
v'	longitudinal velocity in plane perpendicular to vortex wake
V	flight speed of helicopter
x, y, z	coordinates in rotor plane
$\alpha,$	rotor disc incidence
ξ, η	elliptic coordinates in plane perpendicular to vortex wake
ζ	$\xi + i\eta$
ρ	air density
φ	potential function
Φ	acceleration potential
χ	wake angle
ψ	azimuth angle in rotor plane; stream function

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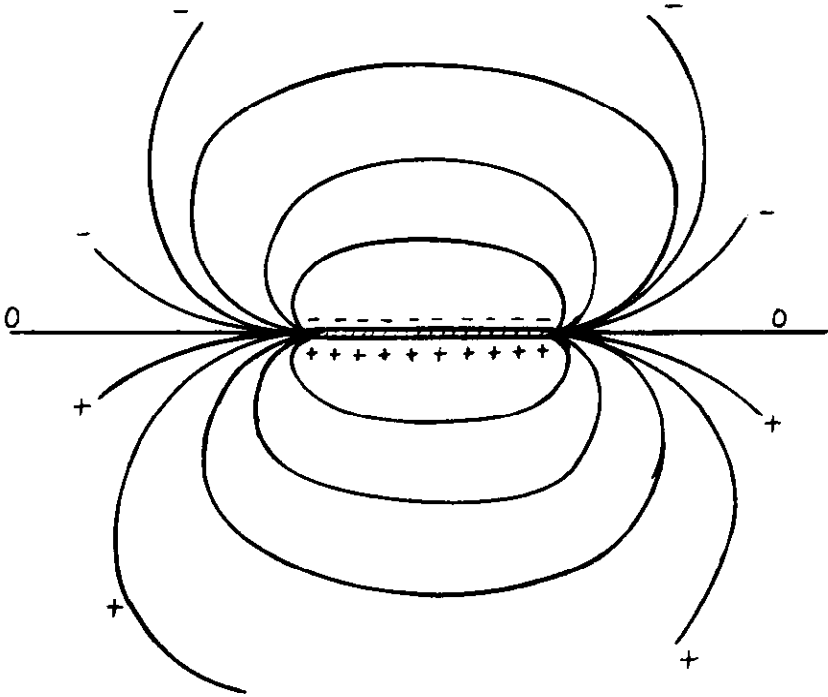
Fig.1 Sketch of pressure field for uniform load
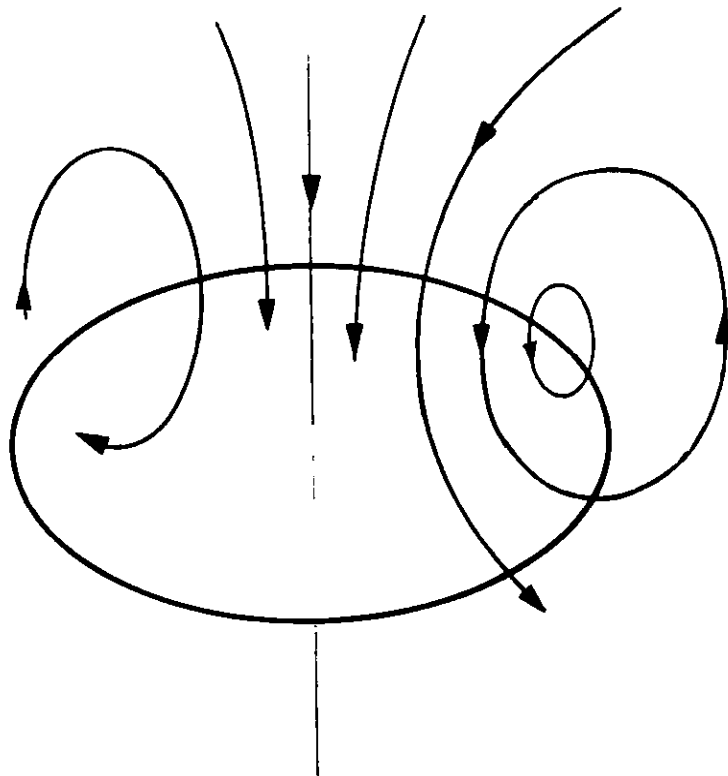
Fig. 2   Sketch of vortex ring flow
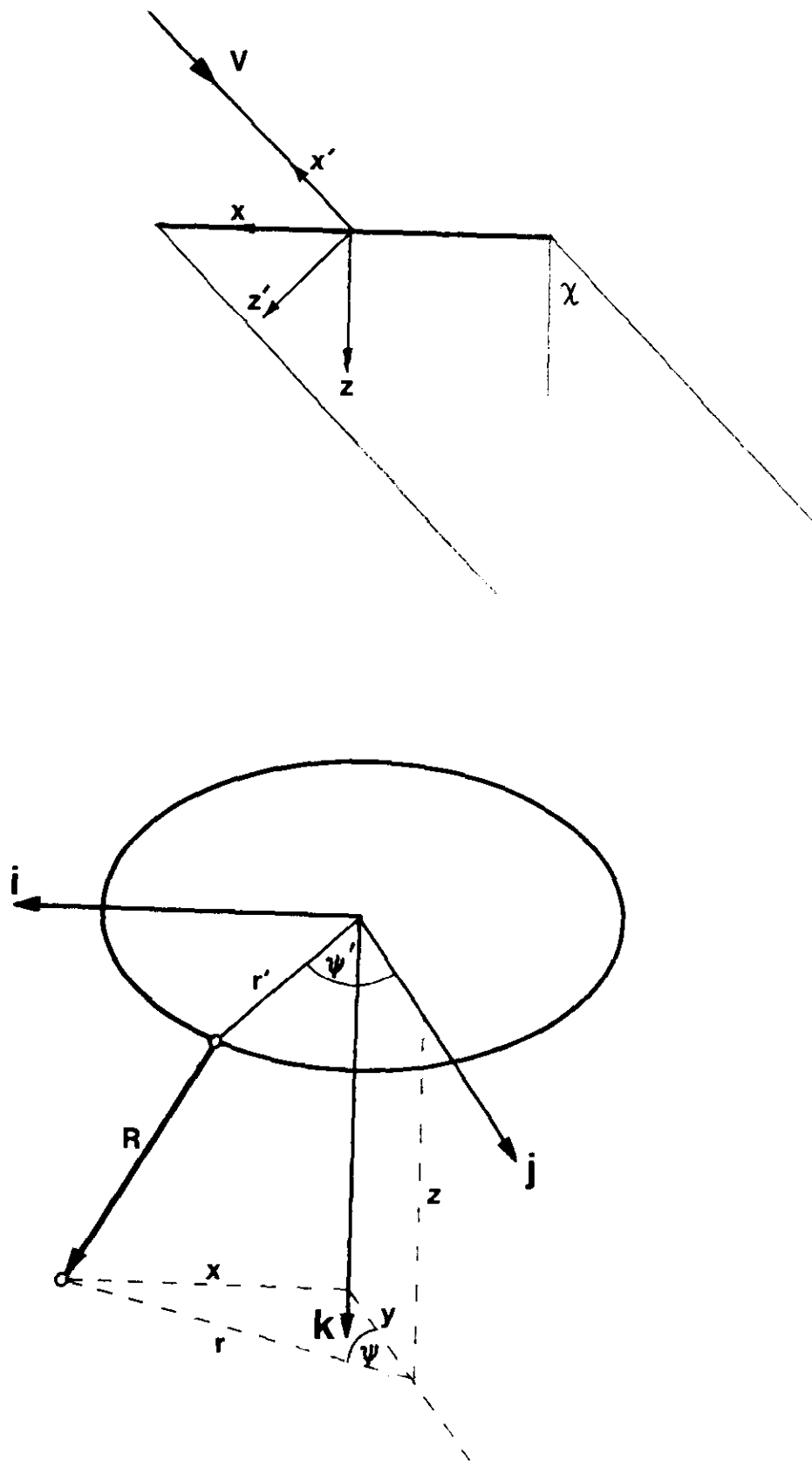
Fig.3   Rotor Coordinates
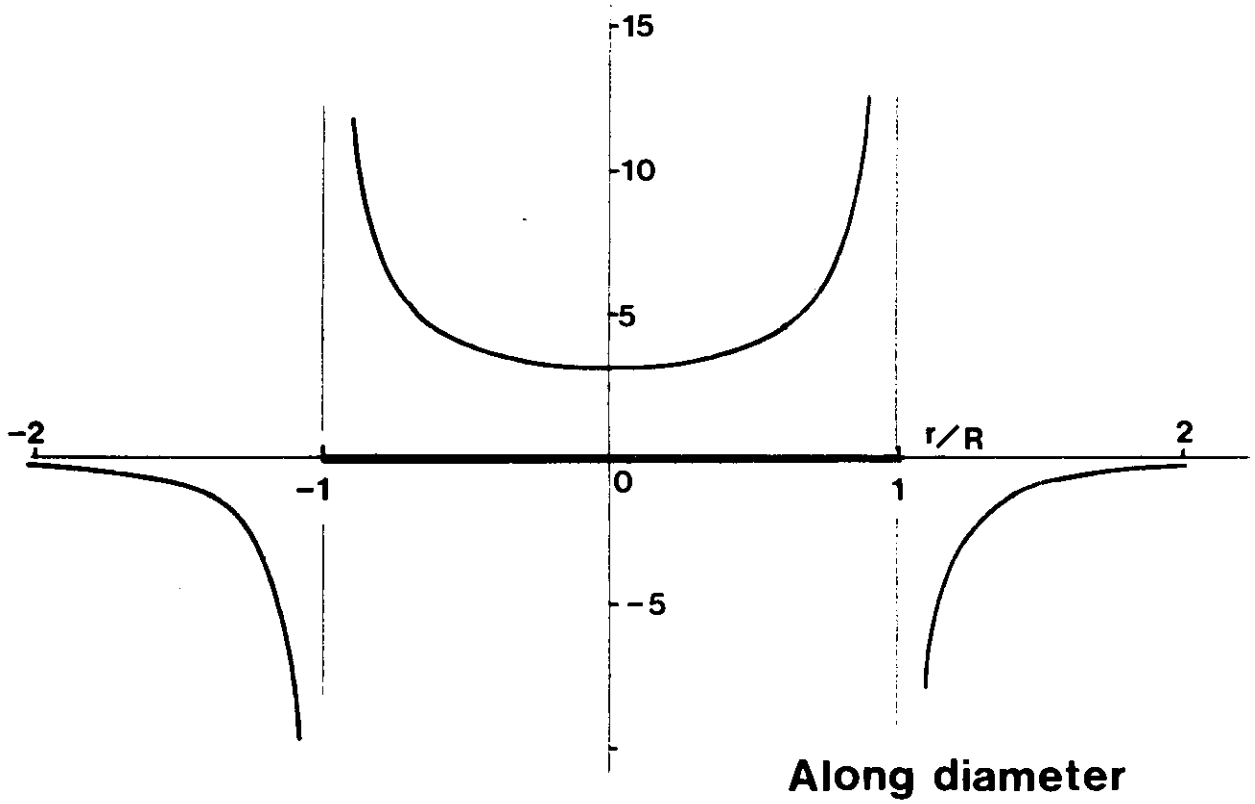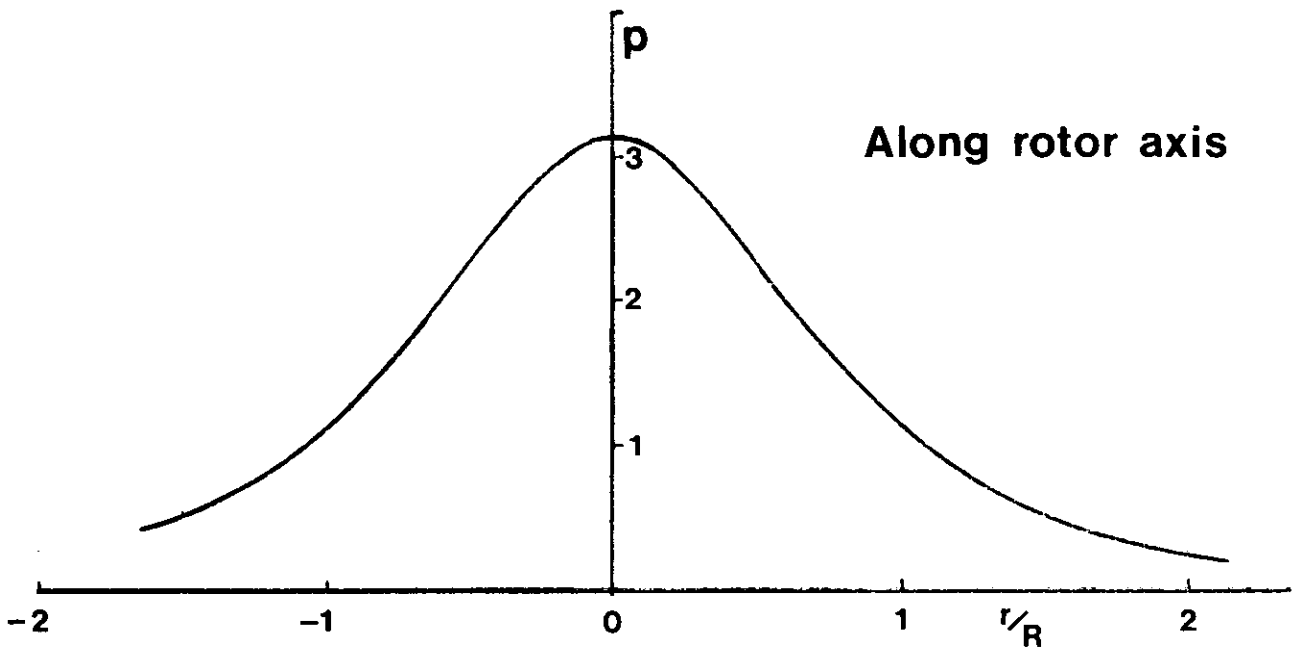
Fig.4   Pressure gradient normal to rotor
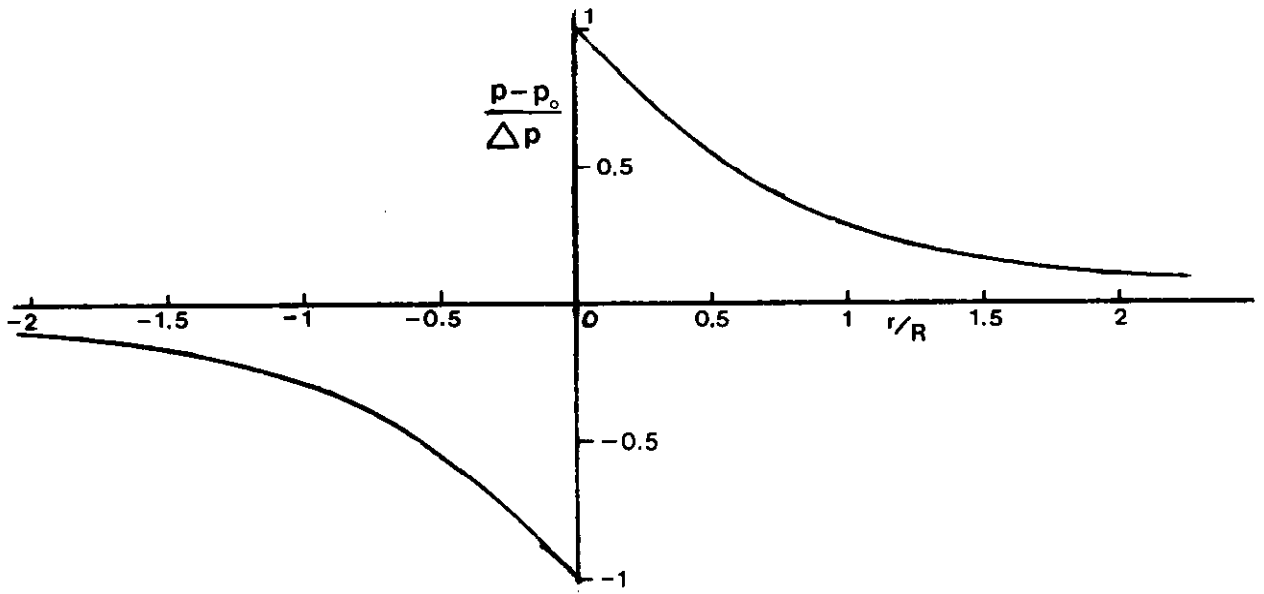
Fig.5   Pressure variation along rotor axis
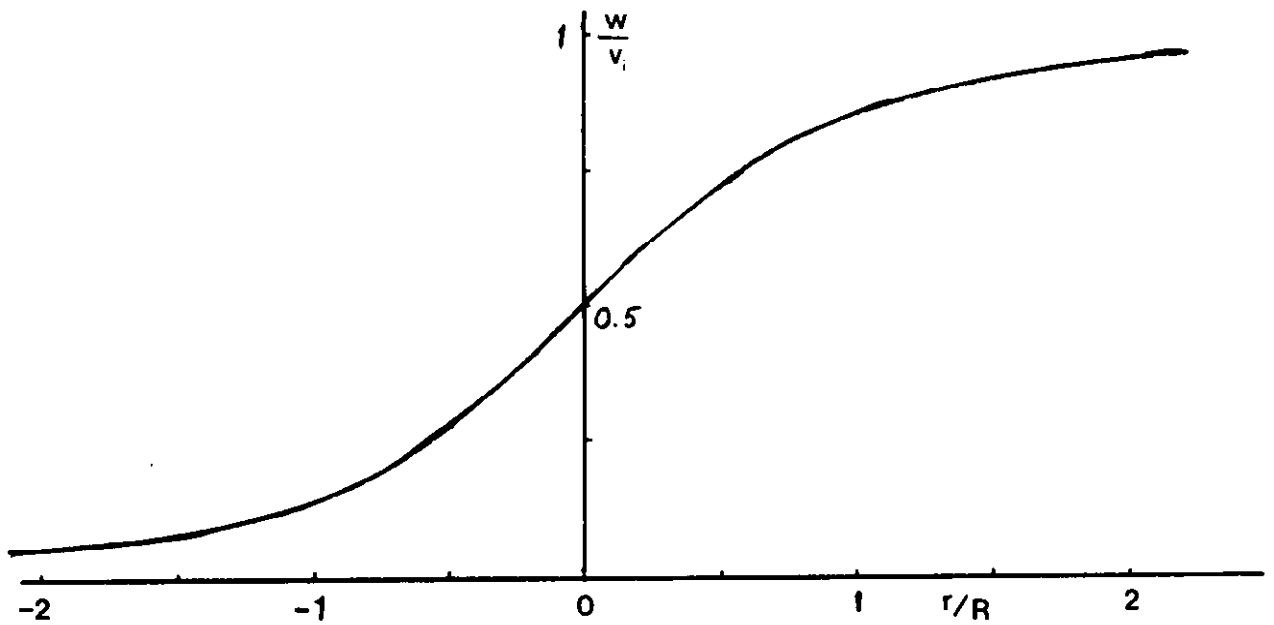
Fig.6 Velocity variation along rotor axis
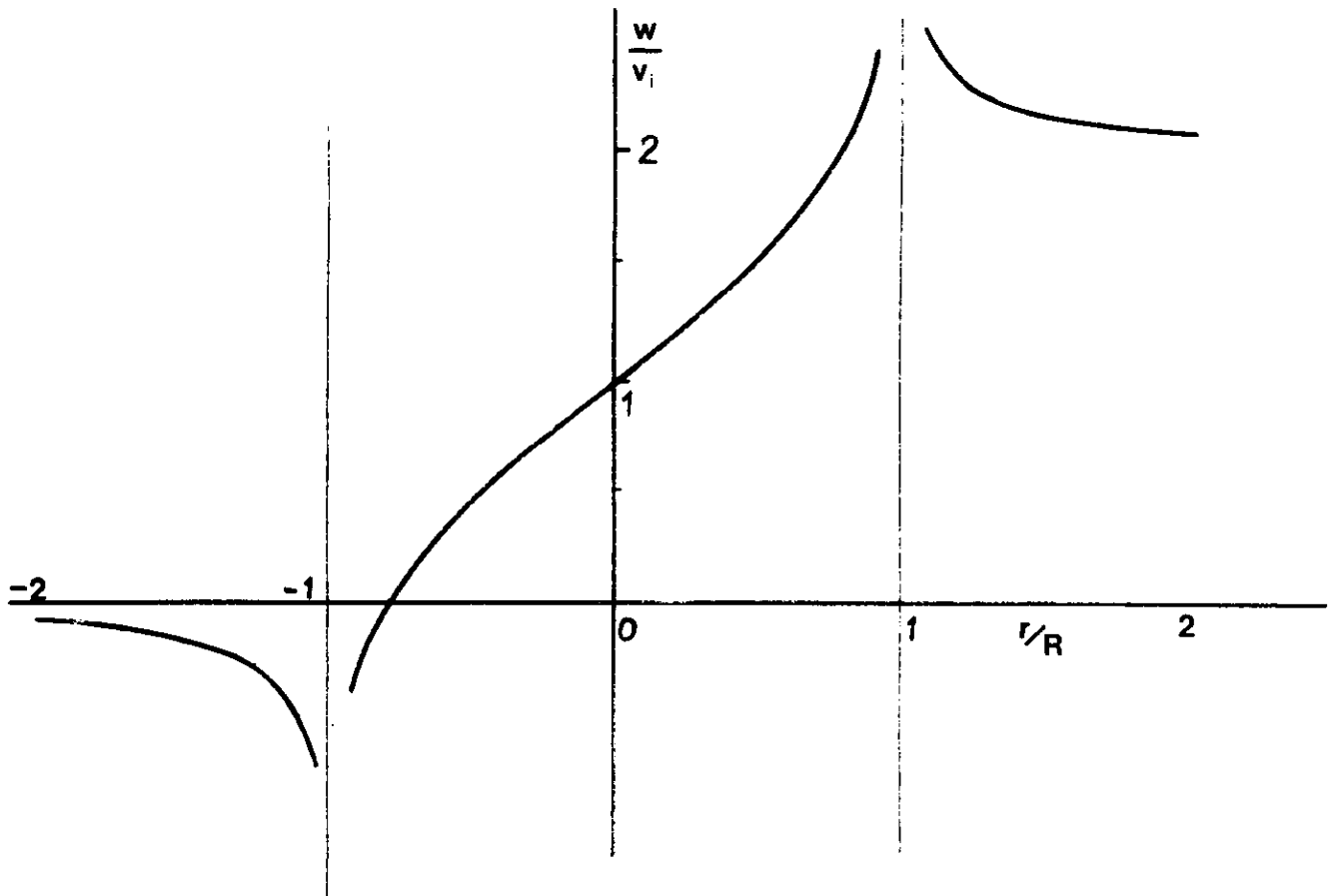
Fig.7 Induced Velocity along longitudinal axis
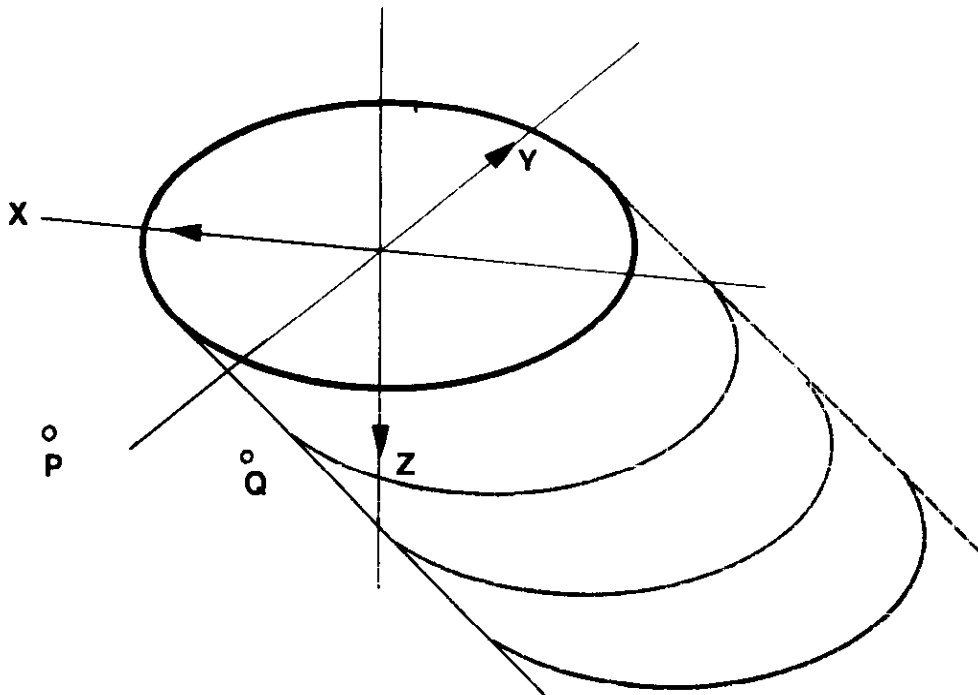
Fig.8 Symmetrical points in wake
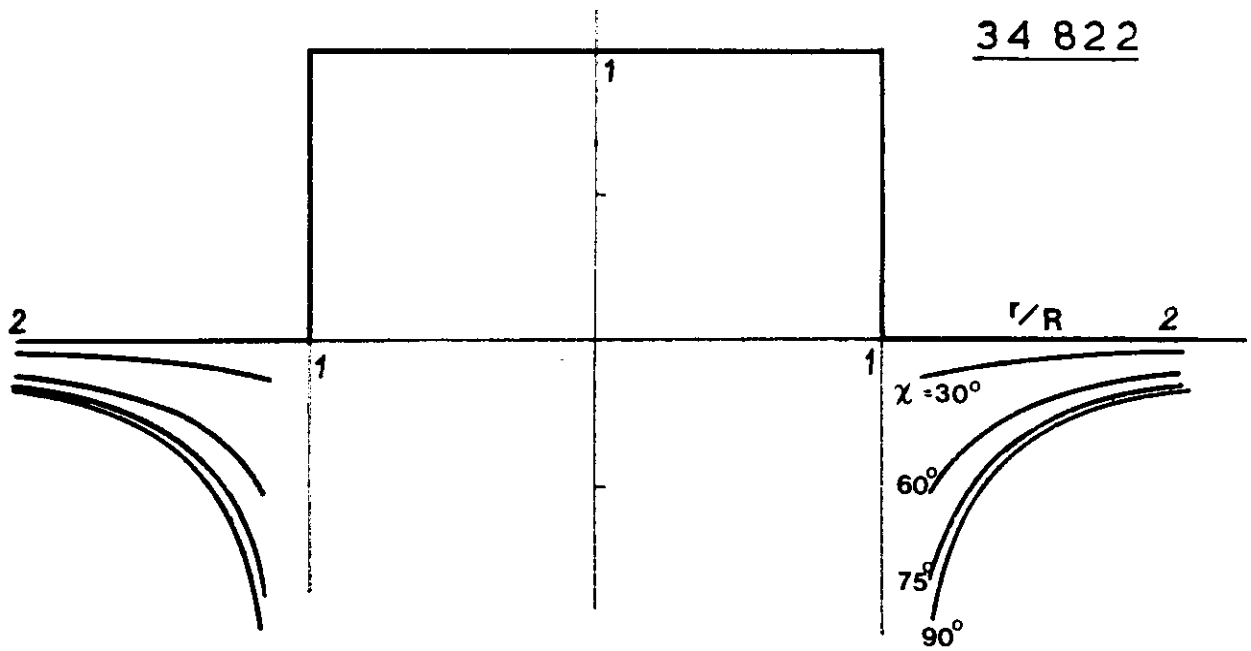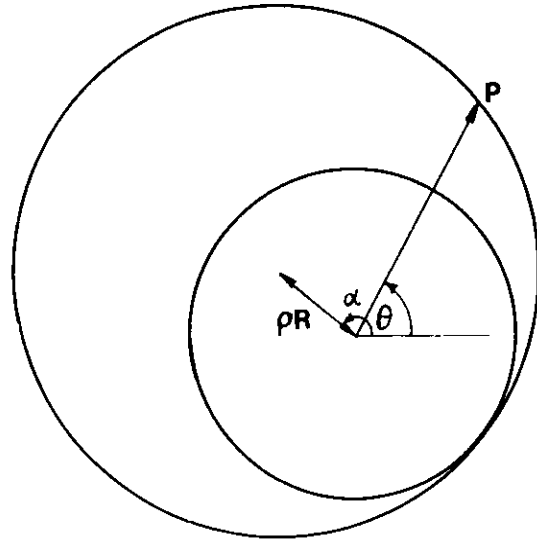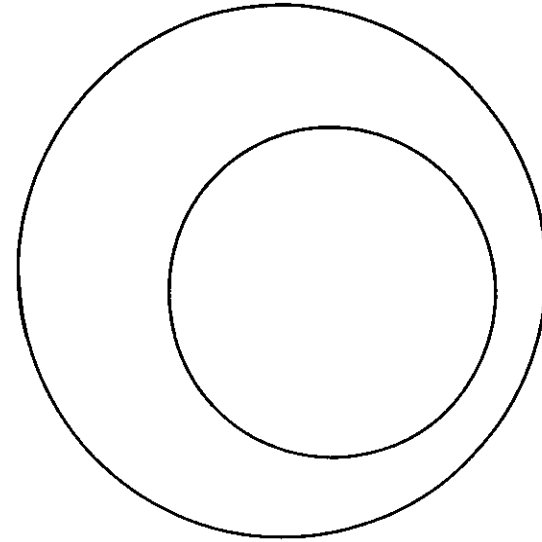
Fig.9   Induced   Velocity   along   lateral   axis
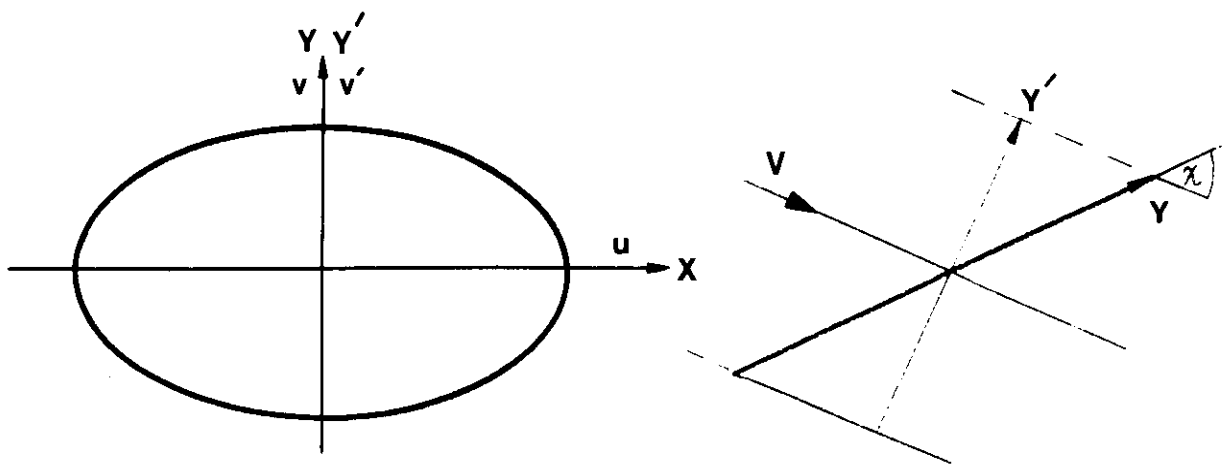
Fig.10   Wake   Coordinates
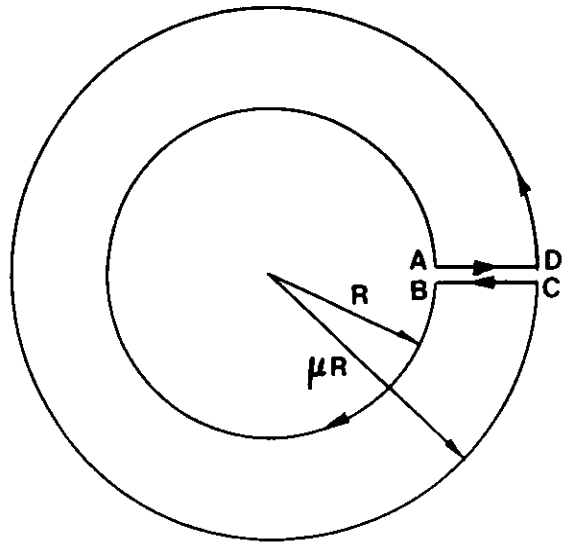
11a   11b   11c

Fig.11   Regions   of   Integration
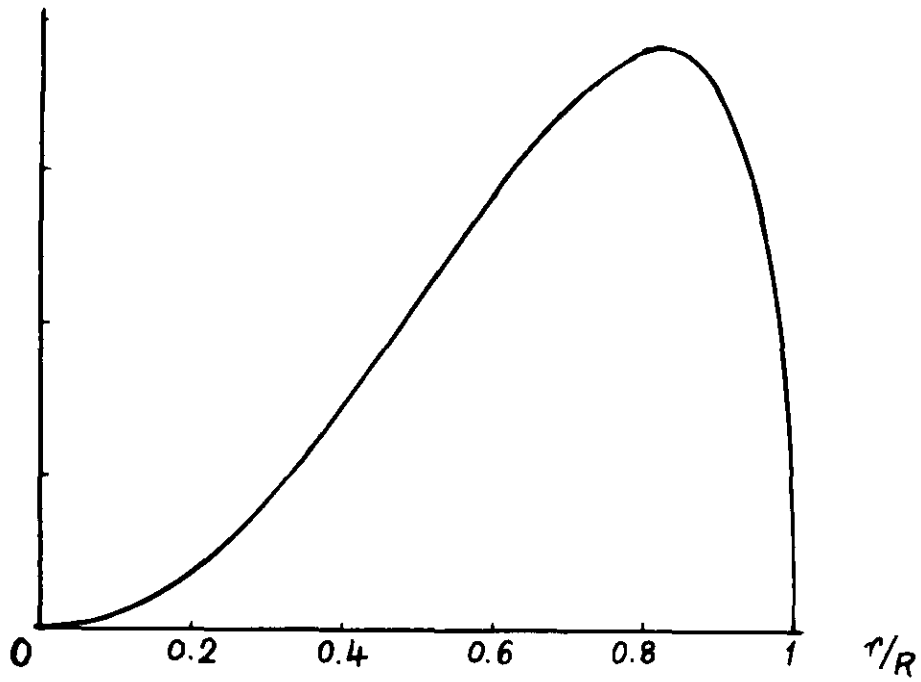
Fig. 12   Mangler's Pressure Distribution
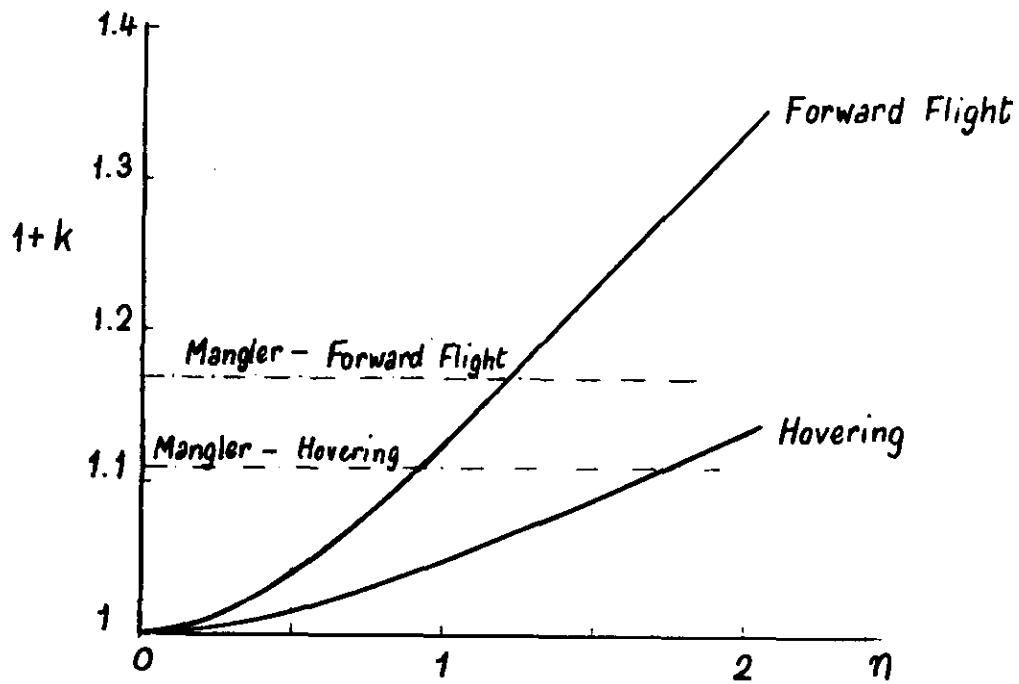
Fig. 13   Induced Power when Rotor Loading is proportional
proportional to r^n

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Bramwell, A. R. S.

SOME REMARKS ON THE INDUCED VELOCITY
FIELD OF A LIFTING ROTOR AND ON GLAUERT'S FORMULA

The induced velocity field of a lifting rotor is
discussed in relation to the pressure field rather than
the vortex wake in an attempt to obtain a clearer
understanding of the relationship between the induced
velocity and rotor forces. A number of results are
derived and, where appropriate, are compared with those
obtained from the theory of the vortex wake.

An/

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formula indicates that it appears to be true for all
rotor loadings for the linearized, "high speed", case.

Calculations of the induced power in forward flight
shows that for typical rotor loadings the power is 40 to
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DETACHABLE ABSTRACT CARDS