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by

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LONDON: HER MAJESTY'S STATIONERY OFFICE

1974

PRICE 75p NET

C.P. No.1290*
December 1972

A SHOCK CAPTURING METHOD FOR CALCULATING SUPERSONIC FLOW FIELDS

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SUMMARY

A finite difference representation of equations that govern the steady supersonic three-dimensional flow of an inviscid ideal gas with constant specific heats is described. A method of using the finite difference equations to calculate solutions of mixed initial and boundary value problems associated with supersonic flow fields is explained. The application of boundary conditions is straightforward because the finite difference equations are based on a form of the supersonic flow equations which has stream surfaces as independent variables.

The finite difference equations are arranged so that shock waves that occur in a flow field are computed automatically. Across computed shock waves, the dependent variables change rapidly but the changes cannot be regarded as discontinuous.

Numerical results for a symmetric delta wing at zero incidence in a uniform mainstream flow are presented. These results are compared with results from a numerical characteristic method.

1. INTRODUCTION

The investigation described in this paper is concerned with the calculation of pressures at points on the surface of an aerodynamic body moving at a steady speed faster than sound. This is an important practical problem which has yet to be solved satisfactorily for the variety of complex shapes which are of interest to designers. Currently much research effort in numerical fluid mechanics is aimed at solving this problem. The work presented here is a contribution to this research.

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*Replaces A.R.C.34 161

In a recent paper, Walkden[1] has derived a special form of the non-linear partial differential equations that govern the steady supersonic three-dimensional flow of an inviscid ideal gas, with constant specific heats. The equations derived in reference (1) are generalisations of some equations for two-dimensional flow which have been used successfully[2],[3] as the basis of a numerical method for calculating supersonic flow fields. In the three-dimensional form of the equations stream-surfaces are used as independent variables.

In this paper we describe a numerical method for calculating three-dimensional supersonic flows. This method is based on the equations of reference (1). The choice of stream surfaces as independent variables simplifies the task of systematically applying boundary conditions for arbitrary body shapes.

For the convenience of readers, the equations of motion derived by Walkden[1] are reproduced in section 2. A method of constructing finite difference approximations to these equations is given in section 3. Section 4 contains a description of the computational procedures to be followed when the finite difference equations have to be solved for a supersonic external flow field generated by an arbitrary body.

The method described in sections 3 and 4 of this paper has been used to calculate the flow past a delta wing which has round subsonic leading edges, which is symmetric with a pointed nose and which is placed at zero incidence in a uniform steady supersonic main-stream flow. The resulting flow is supersonic everywhere. The wing shape and some numerical results are given in 5. The numerical results obtained are discussed in 6. Section 7 contains some concluding remarks.

2. EQUATIONS OF MOTION

The following equations from reference (1) have been used in the investigation described in this paper.

$$
\begin{aligned}
&-\frac{(v')^2 (x^2)^r}{J} t^3_3 \left((J)^2 + \frac{1}{\lambda}(g_{12}g_{33} - g_{13}g_{23}) \right)\left(\frac{\partial\alpha_2}{\partial\xi^2} + \frac{1}{\lambda}\frac{\partial\alpha_2}{\partial\xi^1} \right) \\
&+ \frac{(v')^2}{J} t^2_3 \left((J)^2 + \frac{1}{\lambda}(g_{12}g_{33} - g_{13}g_{23}) \right)\left(\frac{\partial\alpha_3}{\partial\xi^2} + \frac{1}{\lambda}\frac{\partial\alpha_3}{\partial\xi^1} \right) \\
&- \frac{1}{\rho}\left(\frac{(M)^2}{g_{11}}(g_{12}g_{33} - g_{13}g_{23}) + g_{33}/\lambda \right)\left(\frac{\partial p}{\partial\xi^2} + \frac{1}{\lambda}\frac{\partial p}{\partial\xi^1} \right) \\
&+ c_4 \left(\frac{J(M)^2}{g_{11}}(\alpha_2 (x^2)^r t^3_2 - \alpha_3 t^2_2) + \frac{g_{23}}{\lambda} + \frac{g_{13}}{\lambda^2} \right) \\
&+ \frac{d_1}{\rho J}\left((J)^2 + \frac{1}{\lambda}(g_{12}g_{33} - g_{13}g_{23}) \right) \\
&- c^{(1)}_3 \left(\frac{(M)^2}{g_{11}}(g_{12}g_{33} - g_{13}g_{23}) + g_{33}/\lambda \right) = 0
\end{aligned}
\tag{2.1}
$$

$$
c_4 = r(v')^2\alpha_3 \left(t^2_1 t^3_3 - t^2_3 t^3_1 \right) + (1/\rho)\, \partial p/\partial\xi^3 \tag{2.2}
$$

$$
c^{(1)}_3 = r(v')^2\alpha_3 \left(t^2_1 t^3_2 - t^2_2 t^3_1 \right) \tag{2.3}
$$

$$
\begin{aligned}
d_1 = &-r\rho v' t^2_1 \left(t^2_2 t^3_3 - t^2_3 t^3_2 \right) \\
&+ \rho v' (x^2)^r t^3_2\, \partial\alpha_2/\partial\xi^3 - \rho v' t^2_2\, \partial\alpha_3/\partial\xi^3
\end{aligned}
\tag{2.4}
$$

$$
\lambda_{\pm} = \frac{(g_{13}g_{23} - g_{12}g_{33}) \pm \sqrt{\left\{ (g_{13}g_{23} - g_{12}g_{33})^2 - (J)^2(1 - (M)^2/g_{11})(g_{11}g_{33} - (g_{13})^2) \right\}}}{(J)^2(1 - (M)^2/g_{11})} \tag{2.5}
$$

$$
\partial\rho/\partial\xi^1 = (\rho/\gamma p)\, \partial p/\partial\xi^1 \tag{2.6}
$$

$$g_{11} v^1 \partial v^1/\partial \xi^1 + (v^1)^2 \alpha_2 \, \partial \alpha_2/\partial \xi^1 + (v^1)^2 \alpha_3 \, \partial \alpha_3/\partial \xi^1 + (1/\rho)\, \partial p/\partial \xi^1 = 0 \tag{2.7}$$

$$\begin{aligned} g_{13} v^1 \partial v^1/\partial \xi^1 &+ (v^1)^2 t_3^2 \, \partial \alpha_2/\partial \xi^1 + (v^1)^2 (x^2)^\tau t_3^3 \, \partial \alpha_3/\partial \xi^1 \\ &= -(1/\rho)\, \partial p/\partial \xi^3 - \tau (v^1)^2 \alpha_3 (t_1^2 t_3^3 - t_1^3 t_3^2) \end{aligned} \tag{2.8}$$

$$\partial x^2/\partial \xi^1 = \alpha_2 \tag{2.9}$$

$$\partial x^3/\partial \xi^1 = \alpha_3 / (x^2)^\tau \tag{2.10}$$

$$\begin{aligned} J_\rho \, \partial v^1/\partial \xi^1 &+ \rho v^1 \left((x^2)^\tau t_3^3 \, \partial \alpha_2/\partial \xi^2 - t_3^2 \, \partial \alpha_3/\partial \xi^2 \right) + (J \rho v^1/\gamma p)\, \partial p/\partial \xi^1 \\ = -\rho v^1 \Big(&\tau t_1^2 (t_2^2 \, \partial x^3/\partial \xi^3 - t_2^3 \, \partial x^2/\partial \xi^3) \\ &+ t_2^2 \, \partial \alpha_3/\partial \xi^3 - (x^2)^\tau t_2^3 \, \partial \alpha_2/\partial \xi^3 \Big) \end{aligned} \tag{2.11}$$

$$\begin{aligned} g_{12} v^1 \partial v^1/\partial \xi^1 &+ (v^1)^2 t_2^2 \, \partial \alpha_2/\partial \xi^1 + (v^1)^2 (x^2)^\tau t_2^3 \, \partial \alpha_3/\partial \xi^1 + (1/\rho)\, \partial p/\partial \xi^2 \\ &= -\tau (v^1)^2 \alpha_3 \, (t_1^2 t_2^3 - t_2^2 t_1^3) \end{aligned} \tag{2.12}$$

Equations (2.2) - (2.4) define certain terms which appear in (2.1). Equation (2.5) defines two characteristic slopes λ_+ and λ_-. The quantity λ_+ is the slope of the right hand characteristic curve and the characteristic relation associated with this curve is obtained by substituting $\lambda = \lambda_+$ in (2.1). Similarly λ_- is the slope of the left hand characteristic curve and substituting $\lambda = \lambda_-$ in equation (2.1) yields the associated characteristic relation.

Equations (2.6), (2.7) - (2.10) and the two equations obtained from (2.1) by substituting $\lambda = \lambda_+$ and $\lambda = \lambda_-$ obtained from (2.5) form a system of seven equations relating the unknowns $p, \rho, v^1, \alpha_2, \alpha_3, x^2$ and x^3.

The independent variables ξ^1, ξ^2 and ξ^3 define surfaces in (ξ^1, ξ^2, ξ^3) space. When the parameter r is zero (x^1, x^2, x^3) are co-ordinates in a cartesian space and when r is equal to 1 (x^1, x^2, x^3) are cylindrical polar co-ordinates. The planes $\xi^2 =$ constant and $\xi^3 =$ constant in (ξ^1, ξ^2, ξ^3) space define surfaces $\xi^2(x^1, x^2, x^3) =$ constant and $\xi^3(x^1, x^2, x^3) =$ constant in (x^1, x^2, x^3) space. For the equations given in this section, these surfaces are stream surfaces in (x^1, x^2, x^3) space. It must be mentioned also that $\xi^1 = x^1$ in the equations of this section. For further details the reader should consult reference (1).

Now

$$t^i_j = \partial x^i / \partial \xi^j \qquad (i, j = 1, 2, 3) \tag{2.13}$$

$$g_{11} = 1 + (\alpha_2)^2 + (\alpha_3)^2 \tag{2.14}$$

$$g_{12} = \alpha_2 t^2_2 + \alpha_3 (x^2)^r t^3_2 \tag{2.15}$$

$$g_{13} = \alpha_2 t^2_3 + \alpha_3 (x^2)^r t^3_3 \tag{2.16}$$

$$g_{ij} = t^2_i t^2_j + (x^2)^{2r} t^3_i t^3_j \qquad (i, j = 2, 3) \tag{2.17}$$

and

$$J = (x^2)^r (t^2_2 t^3_3 - t^2_3 t^3_2) \tag{2.18}$$

The quantities t^i_j are elements in a transformation from (ξ^1, ξ^2, ξ^3) space to (x^1, x^2, x^3) space. J is the Jacobian determinant of the transformation and the elements g_{ij} $(i, j = 1, 2, 3)$ are elements of the metric tensor in (ξ^1, ξ^2, ξ^3) space. The quantity V^1 is a contravariant velocity component. The other two contravariant velocity components V^2 and V^3 are zero because the surfaces $\xi^2 =$ constant and $\xi^3 =$ constant have been chosen as stream surfaces. It follows that the velocity magnitude

$q = \sqrt{g_{11}} v^1$. It must be stated also that at each point in space the vector with components $(1, \alpha_2, \alpha_3)$ in the directions of the axes of co-ordinates in (x^1, x^2, x^3) space is tangential to the streamline through the point.

Equation (2.11) is a representation of the conservation of mass equation and (2.12) expresses conservation of momentum in the ξ^2 direction. With certain other equations of motion, (2.11) and (2.12) were used in reference (1) to construct the characteristic equations which here we obtain by substituting $\lambda = \lambda_+$ and $\lambda = \lambda_-$ in equation (2.1).

3. FINITE DIFFERENCE EQUATIONS

It will be assumed that a uniform finite difference mesh is constructed in (ξ^1, ξ^2, ξ^3) space. We shall suppose that mesh points in the (ξ^1, ξ^2, ξ^3) directions are separated by constant distances $\Delta\xi^1$, $\Delta\xi^2$ and $\Delta\xi^3$ respectively. We shall suppose further that a boundary condition is to be applied on $\xi^2 = 0$, and that the solution of the equations of motion is required only in some part of the region $\xi^2 \geqslant 0$.

In section 3.1, we list the finite difference equations which have been derived from the equations of motion given in 2. and which have been used to obtain the numerical results shown in 5. Then, in section 3.2, a brief account is given of the way in which the finite difference equations have been constructed.

3.1 Finite Difference Equations for an Arbitrary Mesh Point

In this section finite difference equations for an arbitrary mesh point (i, j, k) in (ξ^1, ξ^2, ξ^3) space are listed. Seven equations are associated with each point (i, j, k). These equations have the form

$$\underline{a} = \underline{b} + \Delta\xi^1 \underline{d} \tag{3.1}$$

where

$$\underline{a} = (a_\beta) \; , \; \underline{b} = (b_\beta) \; , \; \underline{d} = (d_\beta)$$

are column vectors each of which has seven components.

For the equations of motion given in section 2,

$$\begin{aligned}
a_1 &= (p)_{i+1,j,k} \\
a_2 &= (\rho)_{i+1,j,k} \\
a_3 &= (v^1)_{i+1,j,k} \\
a_4 &= (\alpha_2)_{i+1,j,k} \\
a_5 &= (\alpha_3)_{i+1,j,k} \\
a_6 &= (x^2)_{i+1,j,k} \\
a_7 &= (x^3)_{i+1,j,k}
\end{aligned} \tag{3.2}$$

The notation $(\ )_{i,j,k}$ is used to denote the value of the quantity inside the brackets at the mesh point (i,j,k) .

$$
\begin{aligned}
b_1 &= smo(0,j_1) \\
b_2 &= smo(1,j_1) \\
b_3 &= smo(2,j_1) \\
b_4 &= smo(3,1) \text{ if } j \neq 0 \text{ otherwise } b_4 = y(53) \\
b_5 &= smo(4,j_1) \\
b_6 &= y(55) \\
b_7 &= y(56)
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
d_1 &= (\partial p/\partial \xi^1)_{i,j,k} \\
d_2 &= (\partial \rho/\partial \xi^1)_{i,j,k} \\
d_3 &= (\partial v^1/\partial \xi^1)_{i,j,k} \\
d_4 &= (\partial \alpha_2/\partial \xi^1)_{i,j,k} \\
d_5 &= (\partial \alpha_3/\partial \xi^1)_{i,j,k} \\
d_6 &= (\partial x^2/\partial \xi^1)_{i,j,k} \\
d_7 &= (\partial x^3/\partial \xi^1)_{i,j,k}
\end{aligned}
\tag{3.4}
$$

The elements $y(k)$ for integer k that are used in equation (3.2) are defined in table 1. The quantity $Smo(n, \dot{j}_1)$ for integer n and $\dot{j}_1 =$ 1 or 0 is defined to be

$$Smo(n,\dot{j}_1) = \sigma y(50+n) + 0.5\sigma(1-0.5\dot{j}_1)\left(y(10+n)+y(20+n)+\dot{j}_1\left(y(30+n)+y(40+n)\right)\right) \quad (3.5)$$

The quantity $\dot{j}_1$ is set equal to unity if $\dot{j} \neq 0$ and it is set equal to zero if $\dot{j} = 0$. The quantity σ is a pre-assigned constant.

It is clear from equations (3.3) and (3.5) that the quantities b_α, $\alpha = 1, 2, \dots 7$ are functions of the dependent variables at the five mesh points $(\dot{i}, \dot{j}, k)$ $(\dot{i}, \dot{j}\pm 1, k)$ and $(\dot{i}, \dot{j}, k\pm 1)$. Relationships which can be used to determine d_α, $\alpha = 1, 2, \dots 7$, in terms of values of the dependent variables at these points will be given later. We can state now, however, that if the values of the dependent variables are known at the five mesh points $(\dot{i}, \dot{j}, k)$ $(\dot{i}, \dot{j}\pm 1, k)$, $(\dot{i}, \dot{j}, k\pm 1)$ then their values at the point $(\dot{i}+1, \dot{j}, k)$ can be calculated from equation (3.1).

The evaluation of dependent variables at mesh points in successive planes $\xi^1 = n\Delta\xi^1$ from values of the dependent variables in a preceding plane $\xi^1 = (n-1)\Delta\xi^1$, $n = 1, 2, 3 \dots$, forms the basis of the computational procedure described in section 4 for calculating a complete flow field.

From equations (3.3) and (3.5) it can be seen that the expressions for b_α have different forms depending whether $\dot{j}$ is equal to or is different from zero. It will be seen later that certain of the relationships determining the d_α, $\alpha = 1, 2, 3, \dots 7$, differ depending on $\dot{j}$ being equal to or different from zero. This type of dependence on $\dot{j}$ arises because a boundary condition has to be applied on $\xi^2 = 0$. In this case, of course, values of the dependent variables at the points $(\dot{i}, -1, k)$ inside the boundary are not available.

The equations which enable us to express d_α, $\alpha = 1, 2, \cdots 7$, in terms of dependent variables in the plane $\xi^1 = i\Delta\xi^1$ can be written down now. It will be seen that these equations are approximations to the equations of motion given in section 2.

The expressions

$$d_2 = (b_2/\gamma b_1)\, d_1 \tag{3.6}$$

$$d_6 = b_4 \tag{3.7}$$

and

$$d_7 = b_5/(y(55))^{+} \tag{3.8}$$

are derived from equations (2.6), (2.9) and (2.10) respectively.

The quantities d_α for $\alpha = 1, 3, 4$ and 5 are determined from four linear simultaneous equations which can be expressed in the form

$$E\underline{f} = \underline{g} \tag{3.9}$$

where $E = (e_{\alpha\beta})$ is a four-by-four matrix and $\underline{f} = (f_\beta)$ and $\underline{g} = (g_\alpha)$ are both column vectors having four elements.

For the supersonic flow equations given in 2

$$\begin{aligned} f_1 &= d_3 \\ f_2 &= d_4 \\ f_3 &= d_5 \\ f_4 &= d_1 \end{aligned} \tag{3.10}$$

where the d elements in equation (3.10) are defined in (3.4). The descriptions of the elements $e_{\alpha\beta}$ and g_α are complicated. Before writing down expressions for these quantities it is convenient to define a number of other quantities which are useful because they help to relate the finite difference equations of this section to the equations of motion given in section 2.

If, again, $j_1 = 1$ when $j \neq 0$ and $j_1 = 0$ when $j = 0$ then we define

$$t_1^1 = 1.0 \tag{3.11}$$

$$t_1^2 = b_4 \tag{3.12}$$

$$t_1^3 = b_5/(y(55))^r \tag{3.13}$$

$$t_2^1 = 0.0 \tag{3.14}$$

$$v^1 = b_3 \tag{3.15}$$

$$t_2^2 = (y(45) - y(35 - 20(j_1 - 1)))/((j_1+1)\Delta\xi^2) \tag{3.16}$$

$$t_2^3 = (y(46) - y(36 - 20(j_1 - 1)))/((j_1+1)\Delta\xi^2) \tag{3.17}$$

$$t_3^1 = 0.0 \tag{3.18}$$

$$t_3^2 = (y(25) - y(15))/2\Delta\xi^3 \tag{3.19}$$

$$t_3^3 = (y(26) - y(16))/2\Delta\xi^3 \tag{3.20}$$

$$g_{11} = 1 + (b_4)^2 + (b_5)^2 \tag{3.21}$$

$$g_{12} = t_1^2 t_2^2 + (y(55))^{2r} t_1^3 t_2^3 \tag{3.22}$$

$$g_{13} = t_1^2 t_3^2 + (y(55))^{2r} t_1^3 t_3^3 \tag{3.23}$$

$$g_{22} = (t^2_2)^2 + ((y(55))^r t^3_2)^2 \tag{3.24}$$

$$g_{23} = t^2_2 t^2_3 + (y(55))^{2r} t^3_2 t^3_3 \tag{3.25}$$

$$g_{33} = (t^2_3)^2 + ((y(55))^r t^3_3)^2 \tag{3.26}$$

$$c_1 = g_{11} g_{33} - (g_{13})^2 \tag{3.27}$$

$$c_2 = g_{12} g_{33} - g_{23} g_{13} \tag{3.28}$$

$$c_3 = g_{13} g_{22} - g_{12} g_{23} \tag{3.29}$$

$$J = (y(55))^r (t^2_2 t^3_3 - t^2_3 t^3_2) \tag{3.30}$$

$$(M)^2 = g_{11} (v^1)^2 b_2 / \gamma b_1 \tag{3.31}$$

$$\frac{1}{\lambda_+} = \frac{(g_{13} g_{23} - g_{12} g_{33}) - \sqrt{\{(g_{13} g_{23} - g_{12} g_{33})^2 - (J)^2 (1 - (M)^2/g_{11})(g_{11} g_{33} - (g_{13})^2)\}}}{g_{11} g_{33} - (g_{13})^2} \tag{3.32}$$

$$\frac{1}{\lambda_-} = \frac{(g_{13} g_{23} - g_{12} g_{33}) + \sqrt{\{(g_{13} g_{23} - g_{12} g_{33})^2 - (J)^2 (1 - (M)^2/g_{11})(g_{11} g_{33} - (g_{13})^2)\}}}{g_{11} g_{33} - (g_{13})^2} \tag{3.33}$$

$$x_L = 1/\lambda_+ \tag{3.34}$$

$$x_R = 1/\lambda_- \tag{3.35}$$

$$c_3^{(1)} = (v^1)^2 b_5 (t^2_1 t^3_2 - t^2_2 t^3_1) \tag{3.36}$$

$$c_4 = (y(20) - y(10))/(2 b_2 \Delta\xi^3) + \tau (v^1)^2 b_5 (t_1^2 t_3^3 - t_3^2 t_1^3) \tag{3.37}$$

$$\begin{aligned} c_5 = & -\tau b_2 v^1 t_1^2 (t_2^2 t_3^3 - t_3^2 t_2^3) \\ & + b_2 v^1 (y(55))^\tau t_2^3((y(23) - y(13))/2\Delta\xi^3) \\ & - b_2 v^1 t_2^2 ((y(24) - y(14))/2\Delta\xi^3) \end{aligned} \tag{3.38}$$

It can be seen that all the quantities defined in equations (3.11) - (3.38) are functions of the dependent variables at five mesh points in the plane $\xi^1 = i\Delta\xi^1$, the mesh sizes, and other constants of the problem.

Now we shall write down expressions for the elements $e_{\alpha\beta}$ and g_α $(\alpha, \beta = 1, 2, 3, 4)$

$$\begin{aligned} e_{11} &= 0.0 \\ e_{12} &= -(v^1)^2 (y(55))^\tau t_3^3 ((J)^2 + x_L c_2) x_L / J \\ e_{13} &= (v^1)^2 t_3^2 ((J)^2 + x_L c_2) x_L / J \\ e_{14} &= -x_L (g_{33} x_L + (M)^2 c_2 / g_{11}) / b_2 \\ g_1 &= -b_3 c_5 ((J)^2 + x_L c_2) / J b_2 \\ & \quad - c_4 (x_L (g_{13} x_L + g_{23}) - (M)^2 c_3 / g_{11}) \\ & \quad + c_3^{(1)} ((M)^2 c_2 / g_{11} + x_L g_{33}) \\ & \quad - \frac{e_{12}}{x_L} (y(43) - b_4) / \Delta\xi^2 \\ & \quad - \frac{e_{13}}{x_L} (y(44) - b_5) / \Delta\xi^2 \\ & \quad - \frac{e_{14}}{x_L} (y(40) - b_1) / \Delta\xi^2 \end{aligned} \tag{3.39}$$

$$
\begin{aligned}
e_{21} &= 0.0 \\
e_{22} &= -(v')^2 (y(55))^5\, t_3^3 \left((J)^2 + x_R c_2 \right) x_R / J \\
e_{23} &= (v')^2\, t_3^2 \left((J)^2 + x_R c_2 \right) x_R / J \\
e_{24} &= -\left(g_{33} x_R + (M)^2 c_2 / g_{11} \right) x_R / b_2 \\
g_4 &= -b_3 c_5 \left((J)^2 + x_R c_2 \right) / J b_2 \\
&\quad - c_4 \left(x_R (g_{13} x_R + g_{23}) - (M)^2 c_3 / g_{11} \right) \\
&\quad + c_3^{(1)} \left((M)^2 c_2 / g_{11} + x_R g_{33} \right) \\
&\quad - \frac{e_{22}}{x_R} \left(b_4 - y(33) \right) / \Delta\xi^2 \\
&\quad - \frac{e_{23}}{x_R} \left(b_5 - y(34) \right) / \Delta\xi^2 \\
&\quad - \frac{e_{24}}{x_R} \left(b_1 - y(30) \right) / \Delta\xi^2
\end{aligned}
\tag{3.40}
$$

$$
\begin{aligned}
e_{31} &= g_{11} v' \\
e_{32} &= (v')^2 b_4 \\
e_{33} &= (v')^2 b_5 \\
e_{34} &= 1/b_2 \\
g_3 &= 0.0
\end{aligned}
\tag{3.41}
$$

Scheme B

In scheme B the expressions for $e_{\alpha\beta}$ and g_α that are used when $j = 0$ are:-

(i)

$$
\begin{aligned}
e_{11} &= \bar{J}\rho \\
e_{12} &= 0.0 \\
e_{13} &= 0.0 \\
e_{14} &= \bar{J}_\rho v' / \gamma p \\
g_1 &= c_5 - b_2 v' (y(55))^{r} t_3^3 (y(23) - y(53)) / \Delta\xi^2 \\
&\quad + b_2 v' t_3^2 (y(24) - y(54)) / \Delta\xi^2
\end{aligned}
\tag{3.44}
$$

(ii) $e_{2\beta}$, $(\beta = 1, 2, 3, 4)$, g_2 are as defined in (3.43)

(iii) $e_{3\beta}$, $(\beta = 1, 2, 3, 4)$, g_3 are as defined in (3.41)

(iv) $e_{4\beta}$, $(\beta = 1, 2, 3, 4)$, g_4 are as defined in (3.42)

The equations

$$(e_{\alpha\beta})(f_\beta) = (g_\alpha) \qquad (\alpha = 1, 2, 3, 4) \tag{3.45}$$

that are obtained by substituting in (3.45) the expressions for $e_{\alpha\beta}$, f_β and g_α given by equations (3.10) and (3.39) - (3.44) have been derived from (2.1), (with $\lambda = \lambda_+$ and $\lambda = \lambda_-$ in turn), (2.7) and (2.8) respectively.

The equations (3.45) and (3.6) - (3.8) are linear equations which can be solved to obtain expressions for the seven unknowns d_α, $\alpha = 1, 2, \ldots 7$ in terms of the dependent variables at the five mesh points (i, j, k), $(i, j \pm 1, k)$ $(i, j, k \pm 1)$. Then, as we have remarked earlier, substituting these expressions for d_α in equation (3.1) yields expressions for the dependent variables at the point $(i+1, j, k)$ in terms of the above listed mesh points in the plane $\xi' = i\Delta\xi'$.

3.2 Structure of the Finite Difference Equations

The finite difference equations listed in 3.1 have been derived from the equations of motion given in 2 by applying three elementary ideas. These ideas are explained in 3.2.1, 3.2.2 and 3.2.3.

3.2.1 Construction of a system of equations with two independent variables

The seven equations of motion with three independent variables given in 2 are approximated by a set of partial differential equations having only two independent variables. This is achieved by

(i) setting up a family of planes $\xi^3 = k\Delta\xi^3$, $k = 0, \pm 1, \pm 2, \cdots$ in (ξ^1, ξ^2, ξ^3) space

and

(ii) constructing seven partial differential equations associated with each plane $\xi^3 = k\Delta\xi^3$. These equations have ξ^1 and ξ^2 as independent variables and the values of the functions $p, \rho, v^1, \alpha_2, \alpha_3, x^2$ and x^3 in the three planes $\xi^3 = k\Delta\xi^3$ and $\xi^3 = (k \pm 1)\Delta\xi^3$ are the dependent variables. The partial differential equations which are associated with $\xi^3 = k\Delta\xi^3$ and which have ξ^1 and ξ^2 as independent variables are constructed simply by replacing derivatives with respect to ξ^3 in the equations of motion by central difference expressions. For example, for the plane $\xi^3 = k\Delta\xi^3$, $\frac{\partial p}{\partial \xi^3}$ in equation (2.8) is replaced by

$$\left(p_{k+1}(\xi^1, \xi^2) - p_{k-1}(\xi^1, \xi^2)\right)/2\Delta\xi^3$$

Then the remaining dependent variables in the resulting equations of motion are replaced by their values in the plane $\xi^3 = k\Delta\xi^3$ i.e. for $p(\xi^1, \xi^2, \xi^3)$ we substitute $p_k(\xi^1, \xi^2) = p(\xi^1, \xi^2, k\Delta\xi^3)$ and all the other dependent variables are treated in a similar fashion.

If the total number of planes $\xi^3 = k\Delta\xi^3$ is finite and equal to J then we have $7(J-2)$ hyperbolic partial differential equations (the equations will be hyperbolic provided the main stream Mach number is high enough) for $7J$ unknowns. The differential equations have to be supplemented by extra conditions. These conditions may be symmetry or periodicity conditions on the end planes. In the wing calculations described in 5, symmetry conditions are used.

3.2.2 Treatment of the system of equations with two independent variables

The partial differential equations obtained by following the procedure outlined in 3.2.1 are in the characteristic form required for constructing finite difference equations for hyperbolic systems in the manner described by Walkden and Caine[2]. The Walkden and Caine method is based on an original idea due to Courant, Isaacson and Rees[3] in which ξ^2- derivatives in all left hand characteristic equations are represented by backward differences from $\xi^2 = j\Delta\xi^2$. The ξ^2- derivatives in right hand characteristic relations are represented by forward differences from $\xi^2 = j\Delta\xi^2$. All ξ^1- derivatives in the equations are replaced by forward differences.

The equations $(e_{\alpha\beta})(f_\beta) = (g_\alpha)$ for $\alpha = 1$ and $\alpha = 2$ which were established in 3.1 are finite difference approximations of the right and left hand characteristic relations at the point (i, j, k) . The last three terms in the expressions for g_1 and g_2 in (3.39) and (3.40) respectively are ξ^2- derivatives that have been treated using the rule given in the preceding paragraph.

3.2.3 Smoothing effects

As in the Walkden and Caine[2],[3] method for two-dimensional and axi-symmetric supersonic flow, average values of the dependent variables at the point (i, j, k) are used in the finite difference equations. This step is needed in order to obtain acceptable numerical results for shock wave systems.

Equations (3.3) are used to calculate average values of the dependent variables.

4. CALCULATION OF FLOW FIELDS

4.1 Computational Procedure

In this section we describe the procedure which would be followed in order to compute the flow past an arbitrary body shape such as the delta wing described in 5. These flows are such that disturbances to a uniform flow in which the body is placed are confined to a region bounded by a bow shock wave and the body surface. We shall suppose that the disturbed flow field is known on some plane $x^1 =$ constant and that the problem is to calculate the disturbed flow downstream of this plane subject to the application of a boundary condition on the body surface.

In practice, values of the dependent variables of the problem have to be calculated at mesh points throughout some finite region of ξ-space. We shall suppose that this calculation region is defined by the following inequalities:-

$$\xi^1 \geqslant a \tag{4.1}$$

$$0 \leqslant \xi^2 \leqslant c(\xi^1) \tag{4.2}$$

$$0 \leqslant \xi^3 \leqslant b \tag{4.3}$$

where b is some constant and c is a function of ξ^1 which is defined so that the region within which the dependent variables have to be calculated contains the shock wave.

It will be assumed that

(i) all the dependent variables are known or can be calculated on $\xi^1 = a$.

(ii) symmetry or periodicity conditions are available on $\xi^3 = 0$ and $\xi^3 = b$

(iii) a boundary conditon has to be applied at $\xi^2 = 0$

(iv) for $\xi^2 \geq c(\xi^1)$, the dependent variables take known mainstream values

(v) the calculation does not proceed beyond a point where $c(\xi^1) = c_{max}$ where c_{max} is a suitably large pre-assigned constant.

With these assumed conditions, the problem is to calculate the dependent variables at mesh points throughout the region defined by the inequalities (4.1) - (4.3).

Initially, the number of mesh points in the ξ^2- direction $(j_{max}+1)$ and the number in the ξ^3- direction $(k_{max}+1)$ have to be chosen. The integers j_{max} and k_{max} must be large enough to provide the number of mesh points needed for an adequate numerical description of the flow field which is to be calculated. These integers fix the mesh sizes $\Delta\xi^2$ and $\Delta\xi^3$ in the ξ^2 and ξ^3 directions. It is noted that

$$\Delta\xi^2 = c_{max} / j_{max}$$
$$\Delta\xi^3 = b / k_{max}$$

Next, the step length $\Delta\xi^1$ in the ξ^1- direction has to be chosen so that stable numerical results are obtained (see 4.2 and 6.2). In practice, for the present problem, this means simply that the value chosen for $\Delta\xi^1$ must not be too large.

Once values of $\Delta\xi^1, \Delta\xi^2$ and $\Delta\xi^3$ have been assigned, the finite difference equations described in 3.1 can be applied at each mesh point in the plane $\xi^1 = a$ to calculate values of the dependent variables at mesh points in the intersection of the plane $\xi^1 = a + \Delta\xi^1$ and the region defined bv the inequalities (4.1) - (4.3). This process, using the finite difference equations, can be applied repeatedly to calculate values of the dependent variables at mesh points in successive planes $\xi^1 = a + n\Delta\xi^1$, $n = 2, 3, 4, \ldots$

For the delta wing calculations described in 5, $b = \pi/2$ and symmetry conditions are applied on $\xi^3 = 0$ and $\xi^3 = \pi/2$. The symmetry conditions are used to obtain values of the dependent variables at mesh points on $\xi^3 = -\Delta\xi^3$ and on $\xi^3 = \pi/2 + \Delta\xi^3$. Although such mesh points lie outside the calculation region, values of the dependent values are required there in order to calculate values of the dependent variables at neighbouring mesh points on $\xi^3 = 0$ and $\xi^3 = \pi/2$ respectively.

4.2 Numerical Stability

In 4.1 we have outlined an explicit step-by-step procedure for calculating the dependent variables of the problem at mesh points throughout the calculation region defined by the inequalities (4.1) - (4.3).

This integration procedure may be stable for some values of the mesh sizes $(\Delta\xi^1, \Delta\xi^2, \Delta\xi^3)$ and unstable for others, or it could be unstable for all mesh sizes.

The stability of step-by-step numerical procedures for solving non-linear equations generally cannot be examined directly. In practice, approximate methods based on locally linearised forms of the equations are often used in stability investigations. An account of methods of stability analysis for linear finite difference equations is contained in the book by Richtmeyer and Morton[(5)].

For the present problem, however, due to the way in which the finite difference equations have been constructed, even the approximate stability analysis based on linearised equations is not easily carried out. It was decided that the simplest way to proceed would be to assess whether the computational procedure proposed in 3 and 4 is conditionally stable or not by examining numerical results obtained for a particular problem. Thus one object of the investigation of the delta wing flow described in 5 was to obtain numerical evidence concerning the stability characteristics of the finite difference method used.

5. NUMERICAL RESULTS FOR FLOW PAST A DELTA WING

In this section we present some numerical results for the flow field produced when a delta wing is placed in a uniform supersonic stream. The uniform stream Mach number is 3.5 and the wing shape is defined by the equation

$$x^2 = 1/\sqrt{\{(\cos x^3/a)^2 + (\sin x^3/b)^2\}} \tag{5.1}$$

where

$$a = x^1/\sqrt{\{(M_\infty)^2 - 1\}} \tag{5.2}$$

and

$$b = \begin{cases} x^1/\sqrt{\{(M_\infty)^2-1\}} & \text{if } x^1 \le 0.2 \\ x^1\left[1 - \left(\dfrac{x^1-0.2}{0.8x^1}\right)^4\right]/\sqrt{\{(M_\infty)^2-1\}} & \text{if } x^1 > 0.2 \end{cases} \tag{5.3}$$

Here x^1, x^2 and x^3 are cylindrical polar co-ordinates so that in the finite difference equations (see section 3) the parameter $\tau = 1$.

The wing described by equations (5.1) - (5.3) has a pointed nose and a delta plan form. The leading edges are round and sonic. Upstream of the plane $x^1 = 0.2$ the wing is formed by an axi-symmetric cone. Downstream of $x^1 = 0.2$, the cross-sectional shapes perpendicular to the x^1- axis are ellipses whose eccentricity increases with x^1 . The wing trailing edge is a straight line and at the trailing edge the thickness is zero. The wing is placed at zero incidence to the mainstream flow direction and consequently the flow field which has to be calculated is symmetric about $x^3 = 0, \pi/2, \pi$ and $3\pi/2$. The quantity x^3 is the polar angle and the radial planes $x^3 = 0$ and $x^3 = \pi$ contain the wing leading edges.

All the results given in this section are from calculations starting at the plane $x^1 = 0.2$ in which the disturbed flow field is represented by the Taylor-Maccoll[6] solution for the nose cone of the wing.

The graphs in figures 1 - 6 have been prepared from results obtained for the following three cases:-

<u>c1</u>

$$\Delta\xi^1 = 0.03$$
$$\Delta\xi^2 = 0.0319294 \qquad \sigma = 0.25$$
$$\Delta\xi^3 = 0.314192 \qquad N = 1$$

Scheme A (see section 3) treatment of boundary points

<u>c2</u>

$$\Delta\xi^1 = 0.004$$
$$\Delta\xi^2 = 0.0063859 \qquad \sigma = 0.25$$
$$\Delta\xi^3 = 0.157096 \qquad N = 5$$

Scheme A treatment of boundary points

one level 0 sub-division: $j_1^0 = 0,\ j_2^0 = 1,\ k_1^0 = 0,\ k_2^0 = 2$

Division by factor 2 in the ξ^2 and ξ^3- directions

<u>c3</u>

$$\Delta\xi^1 = 0.004$$
$$\Delta\xi^2 = 0.0063859 \qquad \sigma = 0.25$$
$$\Delta\xi^3 = 0.157096 \qquad N = 5$$

Scheme B used to treat boundary points

one level 0 sub-division: $j_1^0 = 0,\ j_2^0 = 1,\ k_1^0 = 0,\ k_2^0 = 2$

Division by factor 2 in the ξ^2 and ξ^3- directions

In paragraphs c1 - c3, σ is the pre-assigned constant smoothing parameter used in (3.5), and N is the number of finite difference mesh intervals between the wing and the shock wave in the starting plane $x^1 = 0{\cdot}2$. Scheme A and Scheme B are two methods of obtaining the dependent variables at mesh points on $\xi^2 = 0$.

Units have been chosen so that the non-dimensional values of the mainstream velocity magnitude density and pressure are $q_\infty = 1, \rho_\infty = 1, p_\infty = V\gamma(M_\infty)^2$ respectively. M_∞ is the mainstream Mach number. In paragraphs c2 and c3, the phrase starting "one level 0 sub-division ..." refers to a device in the computer program which we have used. This device enables us to insert extra mesh points in selected sub-regions of the working space. The working space is called level 0 . With the aid of these insertions we can assess the significance of truncation errors in particular regions. The insertions are made so that the mesh in the selected sub-region of the working space is divided uniformly by some constant factor in the ξ^2 and ξ^3 - directions. The sub-region of the working space which is divided in the cases c3 and c4 is the region covered by level 0 mesh points for which

$$j_1^0 \leq j \leq j_2^0$$
$$k_1^0 \leq k \leq k_2^0$$

Values of $j_1^0, j_2^0, k_1^0,$ and k_2^0 are given in paragraphs c2 and c3.

Figure 9 is a schematic diagram illustrating the level 0 sub-division.

6. DISCUSSION OF RESULTS FOR THE DELTA WING

In this section we discuss, the numerical representation of the bow shock wave, deductions concerning stability of the finite difference method, and the computed results on the wing surface. Reference is made to results that have been obtained for cases different from the cases listed in 5. Graphs of results for these different cases have not been presented because they are not needed for this discussion.

6.1 Bow Shock Wave Results

The bow shock is the only shock wave in the wing flow field. Shock discontinuities are smoothed in the computing process so that changes in the dependent variables across shock waves are continuous. The thickness of computed shock waves can be reduced by reducing the finite difference mesh size.

In problems with three or more independent variables it is often necessary to work with coarse finite difference meshes. In such cases it is desirable to check that the computed shock wave transition layers do not extend to a body surface thus preventing the computation of accurate results on the body surface. From the graphs in figures 1 - 4 it can be seen that, for the wing flow, with the meshes used in the calculations, the shock transition layer does not extend to the body. Moreover, as expected, the computed shock waves propagate outwards from the body with increasing x^1.

6.2 Numerical Stability

With the mesh sizes used in cases $\underline{c}_1 - \underline{c}_3$, for which some results are plotted in figures 1 - 6, there is no sign of any numerical instability. By keeping $\Delta\xi^2$ and $\Delta\xi^3$ constant whilst increasing $\Delta\xi^1$, a value of $\Delta\xi^1$ is reached where the computation process becomes unstable. It is clear that the finite difference method described in 3 and 4 is conditionally stable. The numerical experiments that we have performed do not yield an explicit stability condition, we can state only that stable results can be achieved by making $\Delta\xi'$ sufficiently small.

6.3 Wing Surface Results

Assessing the accuracy of a set of numerical results is not easy. We have compared our numerical results with experimental and theoretical results given by Butler[(4)7]. We have examined the effects of changing mesh sizes. In addition we have examined the effect of changing the computational procedure at the boundary.

From figure 4 it can be seen that our results and Butler's results are in reasonable agreement with the experimental results at $x' = 0 \cdot 417$. Further downstream, however, at $x' = 0 \cdot 683$, it is seen from figure 5 that the agreement is not good. At present we have no idea which, if any, of the results shown in figure 5 are correct.

From tests with different mesh sizes we do not think that the difference between our theoretical results and Butler's theoretical and experimental results can be attributed to simple truncation error in our results.

There are a number of possible explanations for the differences between the results shown in figure 5. The experimental results may be unreliable, Butler has used an approximate theory which may be unreliable in the neighbourhood of $x' = 0 \cdot 683$, and the numerical method that we have used may have introduced numerical errors whose elimination could require fundamental changes in the computational procedure described in 3 and 4.

In an attempt to assess whether our numerical results at $x' = 0 \cdot 683$ contain fundamental errors or not, scheme B (see section 3) was introduced and used to compute the dependent variables on the wing surface. The results obtained were not significantly different from those obtained with scheme A and so this test was inconclusive. It did emerge, however, that in the neighbourhood of the wing leading edge, a region in which it was difficult to obtain satisfactory numerical results, scheme B proved to be a good deal more robust than scheme A. This can be seen from figure 8 where the computing process breaks down at the leading edge when $x^1 = 0{\cdot}66$ in the case c_2 whilst in the case c_3 breakdown does not occur so quicklv. Both breakdowns occur because the streamlines just off the body in the plane $x^3 = 0$ approach the wing leading edge. The distance between the wing leading edge and the streamline becomes so small that approximation errors cause the streamline to cross the body surface. This is a physically unrealistic situation and, of course, the computation process breaks down when it occurs.

7. CONCLUSIONS

From numerical experiments in which the finite-difference method described in 3 and 4 has been used to calculate supersonic flow past the delta wing described in 5 we can state:-

(i) scheme B for calculating dependent variables at mesh points on the wing surface is preferred to scheme A

(ii) the numerical method is stable for sufficiently small values of $\Delta\xi'$

(iii) the method of this paper appears to treat shock waves adequately

(iv) the numerical results for the wing are probably satisfactory for $x' \leq 0{\cdot}417$

(v) further investigations are needed to determine a correct set of results for the wing in the region $x' > 0{\cdot}417$. In this context, it might be helpful to have

(a) wing results obtained by using a third independent numerical method

(b) results gained by applying the method of this paper to different supersonic flow problems

(c) results obtained by using the method of this paper to integrate the differential equations solved by Butler.

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R 74491/1/505376 K4 4/74 P

Mesh point	p	ρ	v^1	α_2	α_3	x^2	x^3	ξ^1	ξ^2	ξ^3
$(i, j, k-1)$	10	11	12	13	14	15	16	17	18	19
$(i, j, k+1)$	20	21	22	23	24	25	26	27	28	29
$(i, j-1, k)$	30	31	32	33	34	35	36	37	38	39
$(i, j+1, k)$	40	41	42	43	44	45	46	47	48	49
(i, j, k)	50	51	52	53	54	55	56	57	58	59

TABLE 1

Table of Locations in y-space that are used to store values of the dependant variables $p, \rho, v^1, \alpha_2, \alpha_3$, x^2, x^3 and the independent variables ξ^1, ξ^2, and ξ^3 for given mesh points.

Example: $y(23) = (\alpha_2)_{i,j,k+1}$

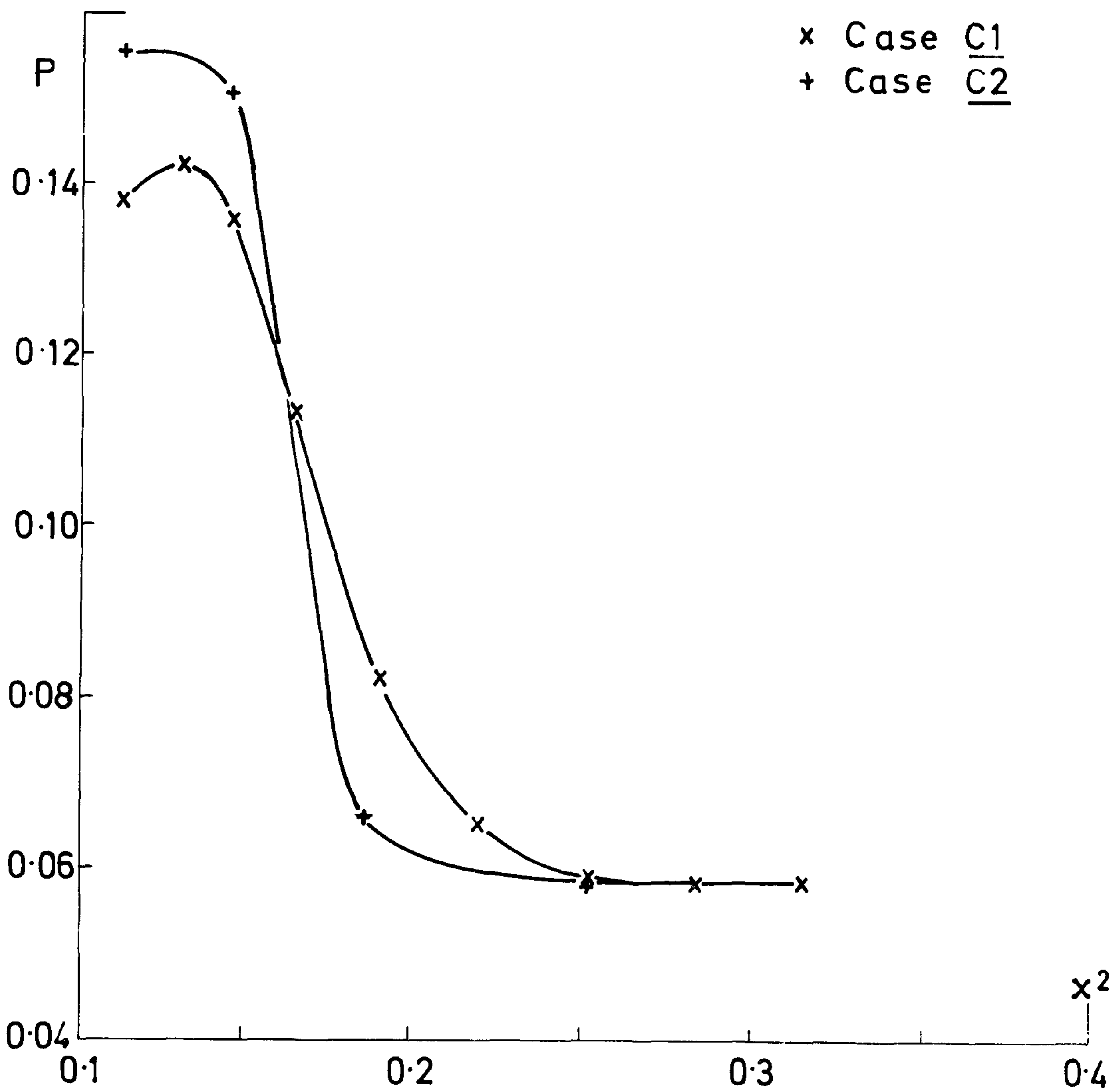

FIG.1. Shock transition layer in the $x^3 = 0$, $x^1 = 0 \cdot 38$

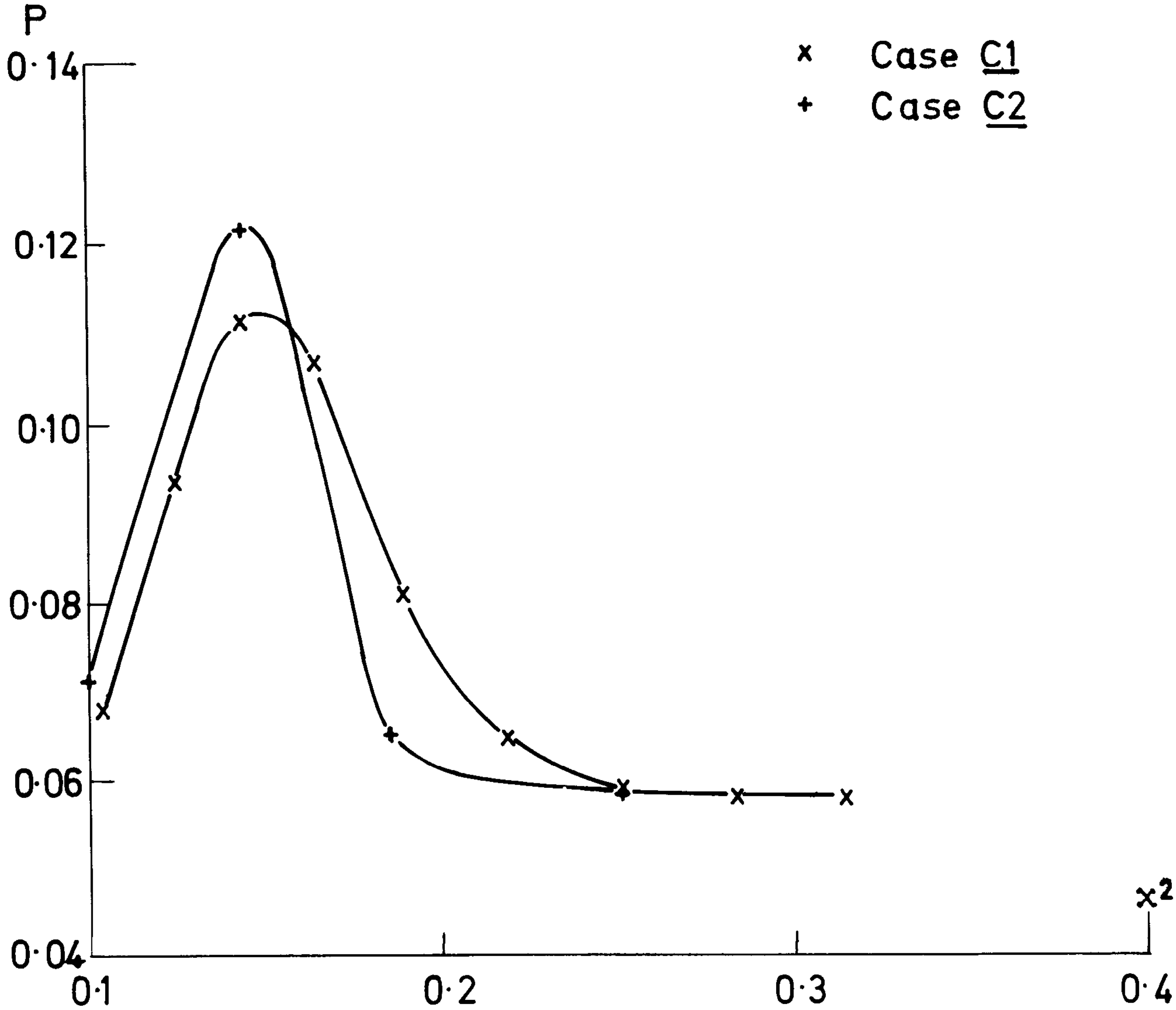

FIG. 2 Shock transition layer in the plane $x^3 = \pi/2$, $x^1 = 0{\cdot}38$

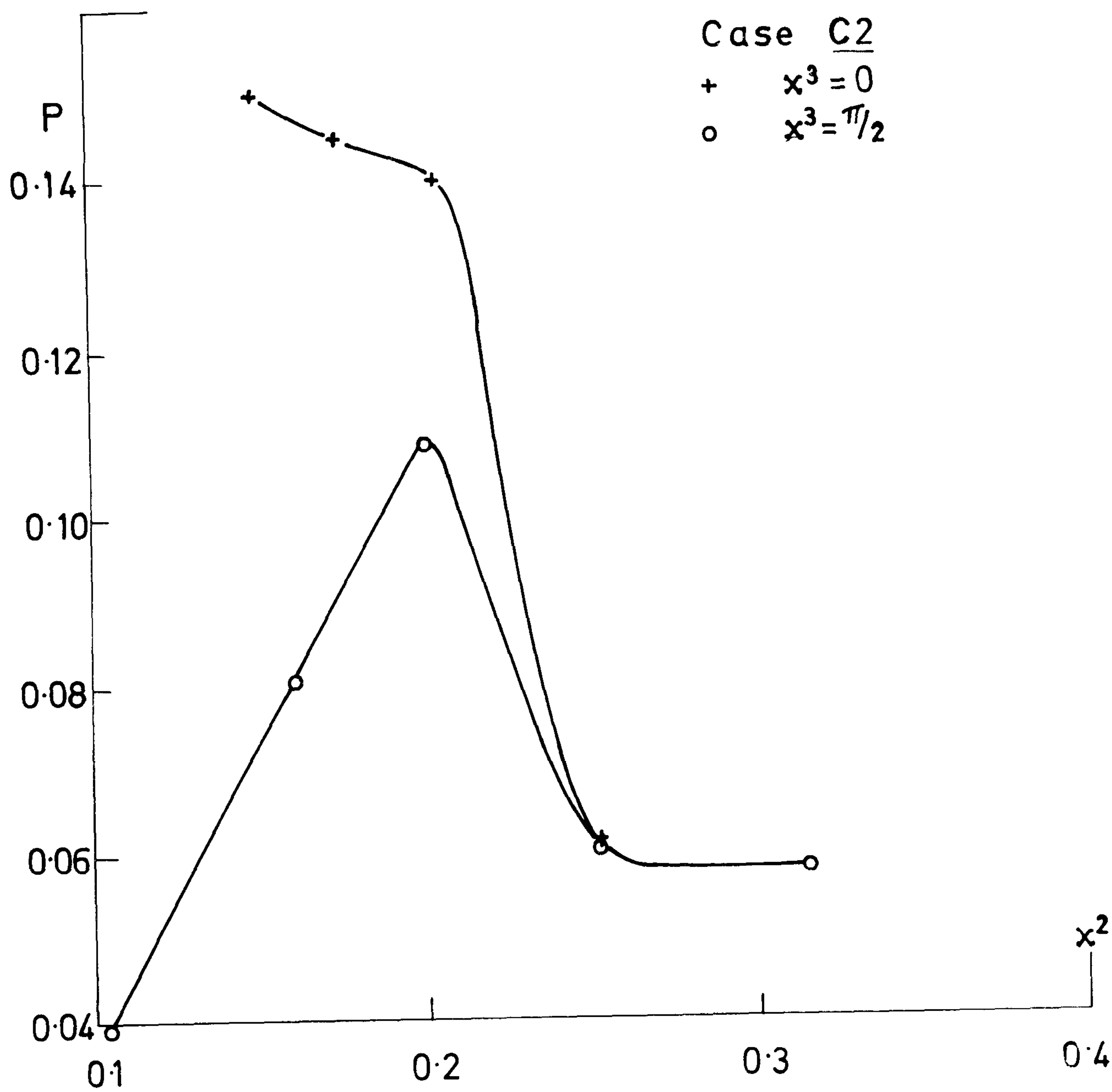

FIG. 3 Shock transition layer in the planes $x^3 = 0$ and $x^3 = \pi/2$, $x^1 = 0 \cdot 504$

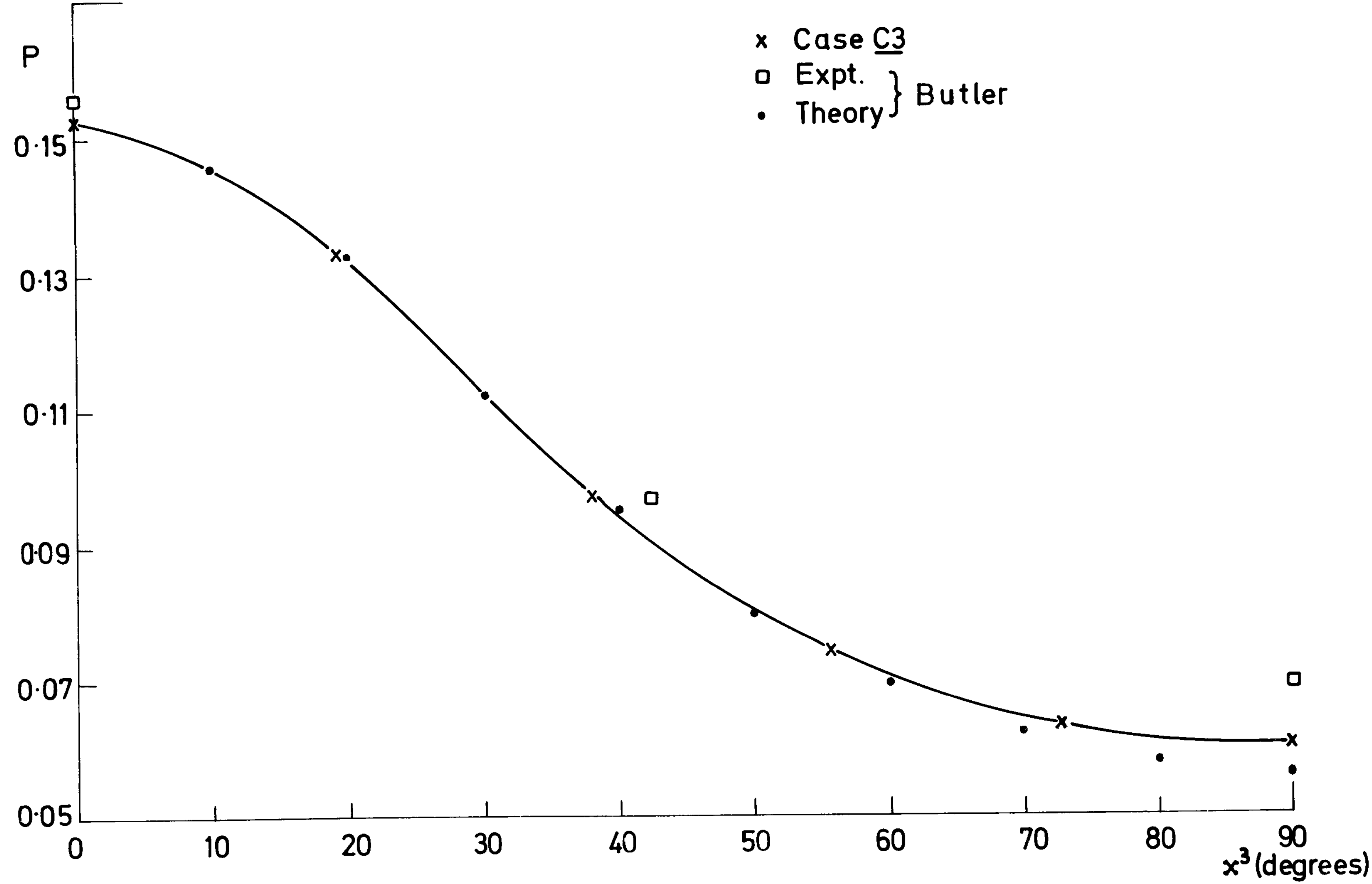

FIG. 4 Surface pressure at $x^1 = 0{\cdot}417$

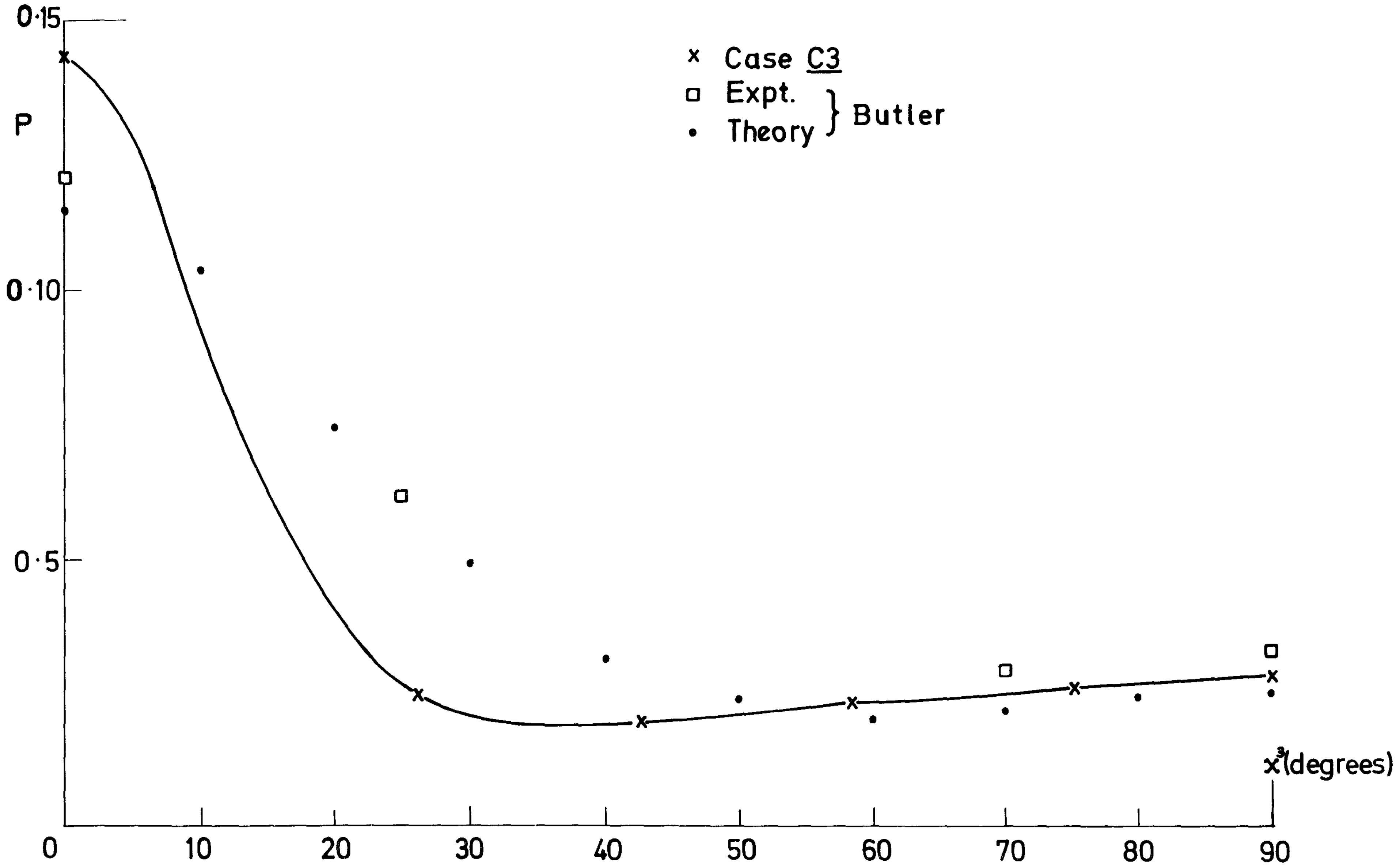

FIG. 5 Surface pressure at $x' = 0 \cdot 683$

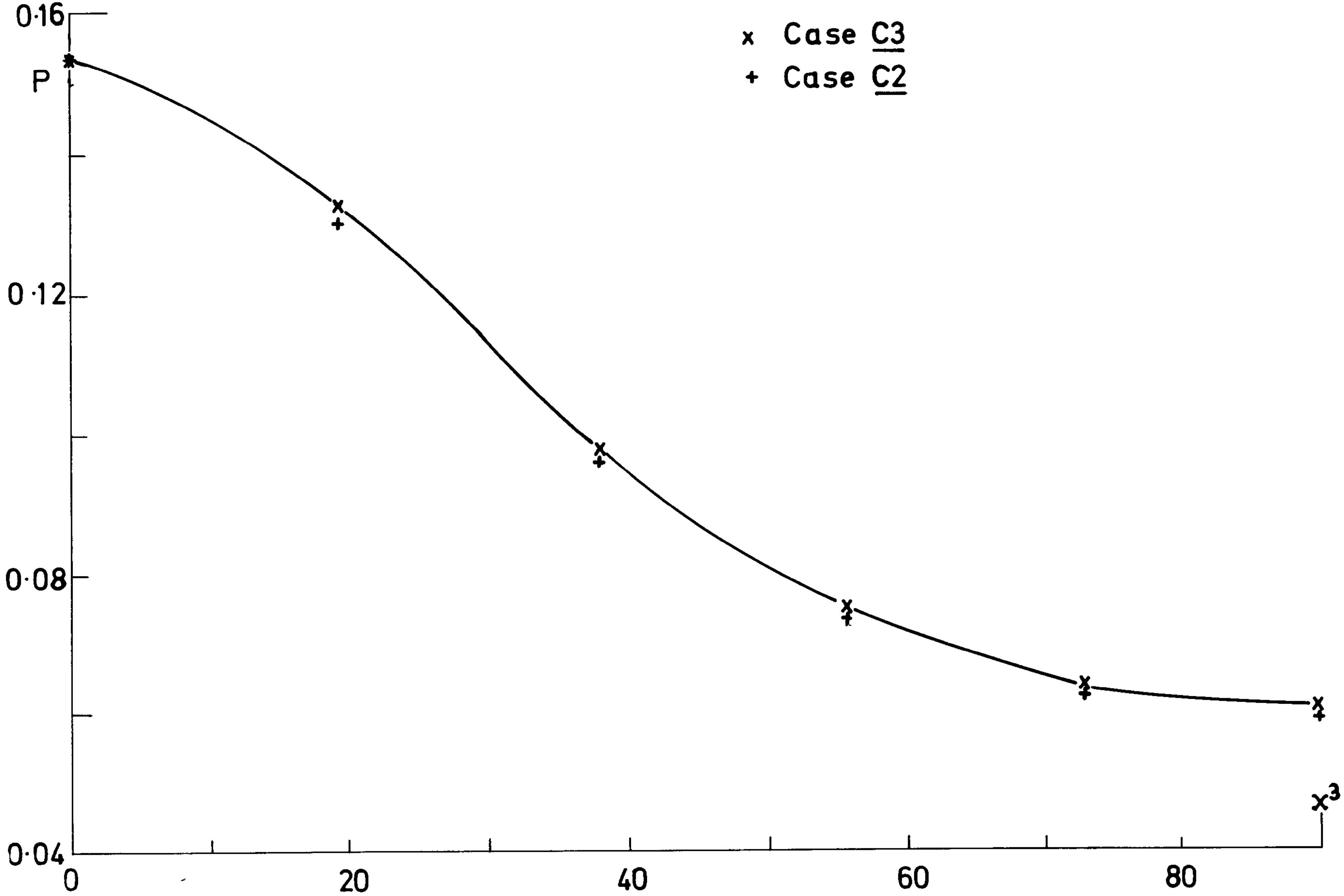

FIG. 6 Scheme A and scheme B, surface pressure at $x^1 = 0 \cdot 417$

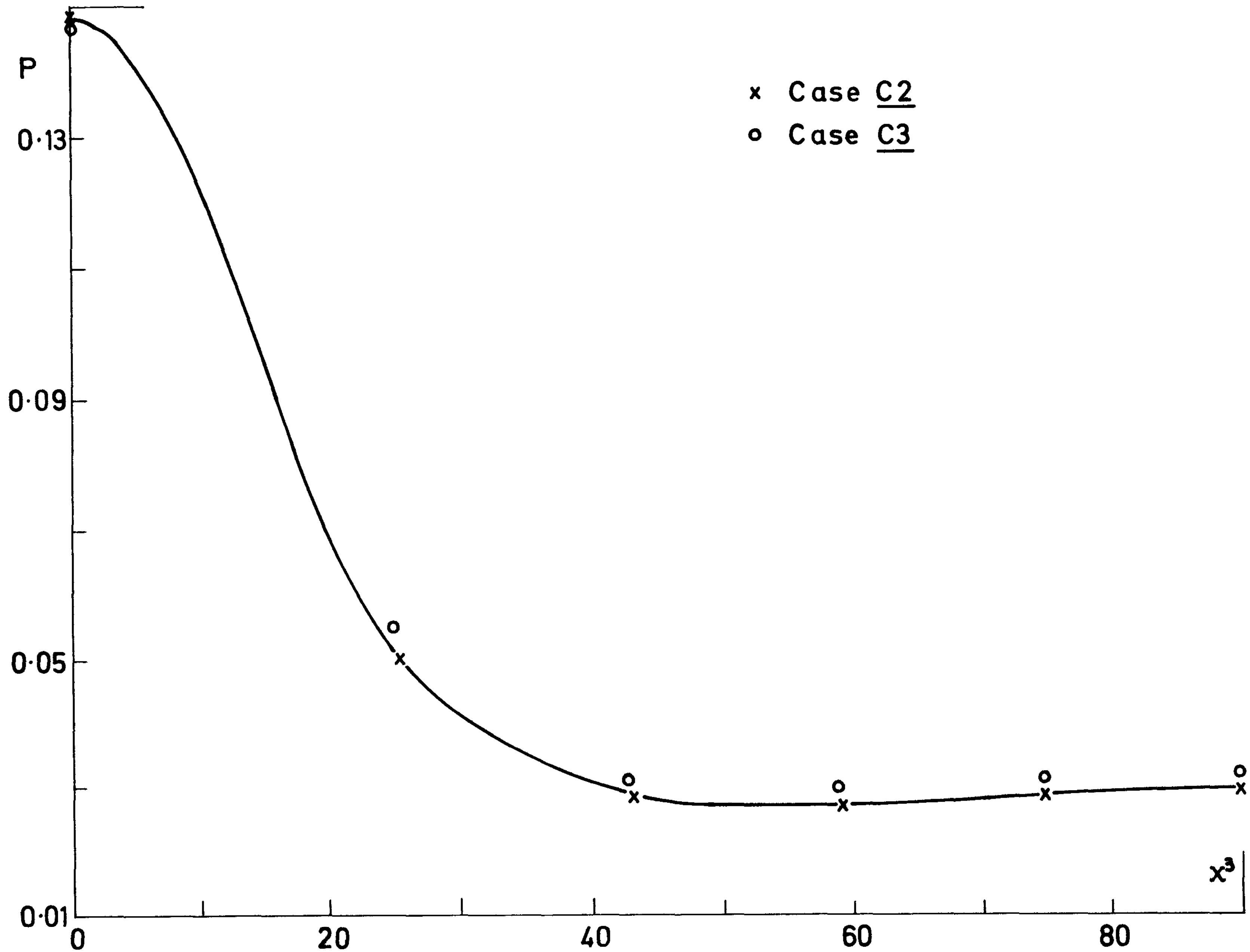

FIG. 7. Scheme A and scheme B, surface pressure at $x^1 = 0{\cdot}61$

ARC CP No.1290
December 1972
Walkden, F. and Caine, P.

A SHOCK CAPTURING METHOD FOR CALCULATING SUPERSONIC FLOW FIELDS

A finite difference representation of equations that govern the steady supersonic three-dimensional flow of an inviscid ideal gas with constant specific heats is described. A method of using the finite difference equations to calculate solutions of mixed initial and boundary value problems associated with supersonic flow fields is explained. The application of boundary conditions is straightforward because the finite difference equations are based on a form of the

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supersonic flow equations which has stream surfaces as independent variables.

The finite difference equations are arranged so that shock waves that occur in a flow field are computed automatically. Across computed shock waves, the dependent variables change rapidly but the changes cannot be regarded as discontinuous.

Numerical results for a symmetric delta wing at zero incidence in a uniform mainstream flow are presented. These results are compared with results from a numerical characteristic method.