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APPLICATION OF A VARIATIONAL METHOD IN PLANE COMPRESSIBLE FLOW CALCULATION

by

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SUMMARY

The classical problem of steady inviscid plane subsonic flow past an aerofoil is formulated as a variational principle, the Bateman-Dirichlet Principle. A finite difference method is used to calculate approximations to the extremals for flow past ellipses and Karman-Trefftz profiles of different thickness ratios. The solutions obtained for the ellipses compare well with other approximate solutions except near the stagnation points where differences of up to 5% are encountered.

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CONTENTS

1 INTRODUCTION

The classical problem of steady inviscid subsonic flow past an aerofoil can be formulated in two ways. The usual formulation is as a boundary value problem consisting of a set of nonlinear partial differential equations and a set of boundary conditions. However, it can also be formulated in terms of complementary variational principles, see e.g. Serrin[1] and Arthurs[2]. There are two such principles which are related in that, for the exact solution, the values of the variational integrals are equal.

Usually procedures for the calculation of approximate solutions are based on the boundary value problem formulation, but it is also possible to use the variational formulation. In this paper we describe a variational method for obtaining numerical solutions. The method consists of replacing the infinitely dimensional variational problem by a finitely dimensional problem by means of finite differences, and an approximate maximizing function is then found by standard methods. A similar approach has been used by Greenspan and Jain[3] for the plane flow past a circle. However, their results near the stagnation points differ greatly from other approximate solutions, so it was felt that it would be advantageous to reconsider the method. The method used in this study differs in some important aspects from that used by Greenspan and Jain[3], mainly in the approximation of the derivatives of the potential by finite differences and in the treatment of the boundary conditions on the aerofoil.

We have obtained results for non-lifting ellipses of different thickness ratios and for different values of the free stream Mach number M_∞. The results are compared with those obtained by Sells[4]. For a circle with M_∞ just below the critical value there is almost complete agreement, the difference is less than 0.5%. Extensive results have also been obtained for a 10% ellipse, and the maximum values of the local Mach number on the surface agree well with those calculated by Sells for different values of M_∞. However, near the stagnation points the difference between the two types of solutions is of the order of 5%. Possible reasons for this discrepancy are discussed but no satisfactory explanation has been found. Results have also been obtained for a Karman-Trefftz profile, and representative values of the local Mach number on the surface are presented.

2 FORMULATION

We shall now formulate the boundary value problem for plane subsonic irrotational flow past an aerofoil. Let (x,y) be a Cartesian coordinate system with velocity vector $\underset{\sim}{u} = (u_1, u_2)$. Far from the aerofoil, C, $\underset{\sim}{u}$ has the form $\underset{\sim}{u} = (U, 0)$ where U is a given constant, see Fig.1.

The flow is supposed to be irrotational, so a velocity potential can be defined by

$$\underset{\sim}{u} = \nabla\phi \ .$$

The pressure and density are denoted by p and ρ, respectively. The speed of sound is defined by

$$c^2 = \frac{dp}{d\rho} \ .$$

We can write p and ρ in the form

$$p = p_0 \left(1 - \frac{q^2}{2\beta c_0^2}\right)^{\alpha} ,$$

$$\rho = \rho_0 \left(1 - \frac{q^2}{2\beta c_0^2}\right)^{\beta}$$

where

$$q^2 = \underset{\sim}{u} \cdot \underset{\sim}{u} , \quad \alpha = \frac{\gamma}{\gamma - 1} , \quad \beta = \frac{1}{\gamma - 1} ,$$

the suffix 0 indicates stagnation values.

It is known that the boundary value problem for ϕ is equivalent to a variational principle (see e.g. Rasmussen[5]). If we let R_1 be the flow region and B the boundary we have from Serrin[1] the Bateman-Dirichlet Principle.

Consider the variational problem of maximising the integral

$$J[\phi] = \int_{R_1} p dV + \int_B \phi h dA \qquad (2.1)$$

(dV = area element, dA = arc length element of B = C) among all subsonic velocities $\underset{\sim}{u} = \nabla\phi$. Then $J[\phi]$ is a maximum if $\nabla \cdot (\rho\underset{\sim}{u}) = 0$ and $\rho\underset{\sim}{u} \cdot \hat{n} = h$ on B. Here the normal mass-flux h is prescribed on B such that

$$\text{outflow} = \int_B h\,dA = 0 \ .$$

It is easily seen that if the flow region R_1 becomes infinite the variational integral (2.1) becomes unbounded. Lush and Cherry[6] showed how the integral should be formulated in order to remove this difficulty, and later Lush[8] wrote it in the form

$$J[\phi] = \iint_\infty [p - p_\infty + p_\infty \nabla\phi_0 . \nabla(\phi - \phi_\infty)]\,dxdy \qquad (2.2)$$

where p_∞ = pressure at infinity,

ρ_∞ = density at infinity

ϕ_∞ = potential for a uniform stream

ϕ_0 = potential for incompressible flow past C.

The class of admissible functions is restricted to functions for which

$$\text{(i)} \quad \frac{\partial\phi}{\partial n} = 0 \quad \text{on } C,$$

$$\text{(ii)} \quad \phi = \phi_\infty + U\chi \qquad (2.3)$$

where $|\chi| \leqslant Kr^{-1}$, $|\nabla\chi| \leqslant Kr^{-2}$
as $r = (x^2 + y^2)^{\frac{1}{2}} \to \infty$ and K is a constant.

In order to make it easier to treat a fairly general class of aerofoils, we shall use a conformal transformation to map the aerofoil C onto the unit circle. Let (r,θ) be a polar coordinate system in the transformed plane, the computation plane, with origin at the centre of the unit circle. Then if we write $z = x + iy$ and $\sigma = r(\cos\theta + i\sin\theta)$, the transform modulus becomes

$$T = \left|\frac{dz}{d\sigma}\right| = (x_r^2 + y_r^2)^{\frac{1}{2}} \ .$$

The Jacobian of the transformation is

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = x_r y_\theta - x_\theta y_r \, .$$

Since the transformation is conformal

$$y_\theta = r x_r \quad \text{and} \quad y_r = -\frac{1}{r} x_\theta \, ,$$

so

$$T^2 = x_r^2 + \frac{1}{r^2} x_\theta^2$$

and

$$J = r\left(x_r^2 + \frac{1}{r^2} x_\theta^2\right) .$$

Hence

$$J = rT^2 \, . \tag{2.4}$$

The coordinates r,θ are orthogonal so the element of length $ds = |dz|$ is given by

$$ds^2 = h_1^2 dr^2 + h_2^2 d\theta^2 \, .$$

Also

$$ds^2 = |dz|^2 = \left|\frac{dz}{d\sigma}\right|^2 |d\sigma|^2$$

$$= T^2 \left(dr^2 + r^2 d\theta^2\right) \, .$$

Therefore

$$h_1 = T \quad \text{and} \quad h_2 = rT$$

so

$$\nabla\phi = \frac{1}{T}\left(\hat{r}\phi_r + \frac{\hat{\theta}}{r}\phi_\theta\right) .$$

Since

$$q^2 = (\nabla\phi)^2 \, ,$$

and

$$\phi = U\,(r \cos\theta + \chi) \, ,$$

we have

$$q^2 = \frac{U^2}{T^2}\left[1 + 2\cos\theta\,\chi_r - \frac{2}{r}\sin\theta\,\chi_\theta + \chi_r^2 + \frac{1}{r^2}\chi_\theta^2\right] . \tag{2.5}$$

Also

$$p = p_0 \left(1 - \frac{q^2}{2\beta c_0^2}\right)^{\alpha}$$

and by using the relation

$$p_0 = \left(\frac{\rho_0}{\rho_\infty}\right)^{\gamma} p_\infty$$

we get after some manipulation

$$p = p_\infty \left[1 + \frac{(\gamma - 1)M_\infty^2}{2T^2}\left(T^2 - 1 - 2\cos\theta\,\chi_r + \frac{2}{r}\sin\theta\,\chi_\theta - \chi_r^2 - \frac{1}{r^2}\chi_\theta^2\right)\right]^{\alpha} \qquad (2.6)$$

where the free stream Mach number M_∞ is defined by

$$M_\infty^2 = \frac{2\beta U^2}{2\beta c_0^2 - U^2} .$$

Now

$$\frac{\gamma p_\infty}{\rho_\infty} = c_0^2 - \frac{U^2}{2\beta} ,$$

so

$$\rho_\infty = \frac{2\beta\gamma}{2\beta c_0^2 - U^2} p_\infty .$$

Thus since

$$\phi_0 = U\left(r + \frac{1}{r}\right)\cos\theta ,$$

$$\rho_\infty \nabla\phi_0 \cdot \nabla(\phi - \phi_\infty) = p_\infty \frac{\gamma M_\infty^2}{T^2}\left[\frac{r^2 - 1}{r^2}\cos\theta\,\chi_r - \frac{r^2 + 1}{r^3}\sin\theta\,\chi_\theta\right] .$$

.... (2.7)

When the expressions (2.4), (2.6) and (2.7) are used in (2.2), we see that the variational integral $J[\phi]$ becomes

$$J[\chi] = p_\infty \int_0^{2\pi} d\theta \int_1^\infty \left\{\left[1 + \frac{(\gamma-1)M_\infty^2}{2T^2}\left(T^2 - 1 - 2\cos\theta\,\chi_r + \frac{2}{r}\sin\theta\,\chi_\theta - \chi_r^2 - \frac{1}{r^2}\chi_\theta^2\right)\right]^\alpha - 1 + \frac{\gamma M_\infty^2}{T^2}\left(\frac{r^2-1}{r^2}\cos\theta\,\chi_r - \frac{r^2+1}{r^3}\sin\theta\,\chi_\theta\right)\right\} rT^2\,dr. \quad \ldots\ (2.8)$$

The boundary conditions on χ are

$$\frac{\partial\chi}{\partial r} = -\cos\theta \quad \text{at } r = 1,$$

$$\chi = 0\left(\frac{1}{r}\right) \quad \text{as } r \to \infty. \quad (2.9)$$

The local Mach number M and the local nondimensional pressure p_L are given by

$$M = M_\infty \frac{q}{U}\left[1 + \tfrac{1}{2}(\gamma-1)M_\infty^2\left(1 - \left(\frac{q}{U}\right)^2\right)\right]^{-\frac{1}{2}}, \quad (2.10)$$

$$p_L = \frac{p}{p_\infty} = \left[1 + \tfrac{1}{2}(\gamma-1)M_\infty^2\left(1 - \left(\frac{q}{U}\right)^2\right)\right]^{\gamma/\gamma-1} \quad (2.11)$$

where q is given by (2.5).

The transform modulus T cannot, in general, be expressed analytically except for very special bodies such as ellipses. Thus if the flow past a realistic aerofoil is desired a numerical evaluation of T must be used. In this paper we mainly consider ellipses for which

$$T^2 = \frac{1}{r^4}\left[(r^2 + \lambda^2)^2 - 4\lambda^2 r^2 \cos^2\theta\right] \quad (2.12)$$

where λ^2 is the following function of τ, the thickness ratio of the ellipse,

$$\lambda^2 = \frac{1-\tau}{1+\tau}. \quad (2.13)$$

3 NUMERICAL METHOD

The object of the calculation is to find for given M_∞ and aerofoil shape a function χ which maximizes $J[\chi]$ as given by (2.8), and satisfies the boundary conditions (2.9). If we only consider nonlifting bodies which are symmetric about the axis $y = 0$, it is only necessary to treat the interval $0 \leqslant \theta \leqslant \pi$. Since the derivatives in both directions are approximated by finite differences, it is necessary to have a finite computation region. This is obtained by replacing the infinite integration limit on r by a finite limit R and insisting that the reduced potential χ satisfies an appropriate condition at $r = R$. The manner in which R is determined is described later. The simplest condition to impose is that χ equals the reduced potential for incompressible flow at $r = R$. A more complicated procedure which involves an asymptotic solution that takes into account the shape of the body is developed in the Appendix. When $R = 20$ or larger the two boundary conditions give results which are identical to within the accuracy of the method, but for smaller values of R the second condition is more accurate.

Thus the variational integral (2.8) reduces to

$$J[\chi] = p_\infty \int_0^\pi d\theta \int_1^R F(r,\theta,\chi_r,\chi_\theta)dr \tag{3.1}$$

where

$$F = \left\{\left[1 + \frac{(\gamma - 1)M_\infty^2}{2T^2}\left(T^2 - 1 - 2\cos\theta\,\chi_r + \frac{2}{r}\sin\theta\,\chi_\theta - \chi_r^2 - \frac{1}{r^2}\chi_\theta^2\right)\right]^\alpha - 1 + \frac{\gamma M_\infty^2}{T^2}\left(\frac{r^2 - 1}{r^2}\cos\theta\,\chi_r - \frac{r^2 + 1}{r^3}\sin\theta\,\chi_\theta\right)\right\} rT^2 \quad . \tag{3.2}$$

The boundary conditions are now

$$\left.\begin{aligned} \frac{\partial\chi}{\partial\theta} &= 0 && \text{at} \quad \theta = 0,\ \pi \\ \frac{\partial\chi}{\partial r} &= -\cos\theta && \text{at} \quad r = 1\ , \\ \chi &= \frac{1}{R}\cos\theta && \text{at} \quad r = R \end{aligned}\right\} \tag{3.3}$$

where the last condition can be replaced by the condition derived in the Appendix.

If r and θ are measured along Cartesian axes, we see that the integration domain is a rectangle, Fig.2. The domain is divided into an irregular mesh given by the intersections of two sets of straight lines which are defined by

$$0 = \theta_1 < \theta_2 < \dots < \theta_m = \pi$$

where $h_i = \theta_{i+1} - \theta_i$

and $1 = r_1 < r_2 < \dots < r_n = R$

where $k_j = r_{j+1} - r_j$.

We can now explain how a value for R is decided on. Let $\sigma = 1/r$ so that the interval $1 \leqslant r < \infty$ is mapped into $1 \leqslant \sigma \leqslant 0$, and divide $[0,1]$ into n equal parts. By use of $r = 1/\sigma$ the interval $1 \leqslant r < \infty$ is then divided into n unequal parts, and we set $R = r_n$; i.e. $R = n$. Thus the value of R depends on the number of mesh points in the radial direction. This procedure is, of course, equivalent to mapping the outside of the unit circle onto the inside using the mapping $r = 1/\sigma$ in order to get a finite computation region as was done by Sells[4].

The grid lines of constant θ map into curves in the physical plane which are clustered around the areas of high curvature on the aerofoil, see Sells[4], p.381. Thus the grid used ensures that there are more points in the regions where the flow varies rapidly than elsewhere.

The infinitely-dimensional variational problem is now replaced by a finitely-dimensional problem. Consider four neighbouring points as shown in Fig.2. The derivatives of χ in the rectangle formed of the grid lines i, $i + i$ and j, $j + 1$ are approximated by finite differences

$$\frac{\partial \chi}{\partial \theta} = \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2h_i} ,$$

$$\frac{\partial \chi}{\partial r} = \frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2k_j} .$$

With these expressions $J[\chi]$ can be approximated for the rectangle by

$$J[\chi] \approx J_{i,j}$$

$$= p_\infty \left\{ \left[1 + \frac{(\gamma - 1)M_\infty^2}{2T^2} \left(T^2 - 1 - 2\cos\theta_1 \frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2k_j} \right. \right. \right.$$

$$+ \frac{2}{r_1}\sin\theta_1 \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2h_i}$$

$$- \left(\frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2k_j} \right)^2$$

$$\left. \left. - \frac{1}{r_1^2} \left(\frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2h_i} \right)^2 \right] \right]^\alpha$$

$$- 1 + \frac{\gamma M_\infty^2}{T^2} \left(\frac{r_1^2 - 1}{r_1^2} \right) \cos\theta_1 \frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2k_j}$$

$$\left. \left. - \frac{r_1^2 + 1}{r_1^3} \sin\theta_1 \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2h_i} \right) \right\} r_1 T^2 h_i k_j \qquad \ldots (3.4)$$

where $\theta_1 = \theta_i + 0.5\, h_i$, $r_1 = r_j + 0.5\, k_j$, and T is evaluated for θ_1 and r_1. Before we can sum the contributions for each rectangle, it is necessary to consider the treatment for the boundary conditions.

One of the four boundary conditions that χ must satisfy creates no problems. From (3.3) we see that at the line $j = n$ χ is prescribed, so no modification is required to the procedure described above. However, at the lines $i = 1$, $i = m$ and $j = 1$ only the normal derivative of χ is prescribed. Let us first consider $i = m$. Here χ_θ must be zero, and in order to approximate this we add an extra line $i = m + 1$ to the mesh such that $h_m = h_{m-1}$, and then set

$$\chi_{m+1} = \chi_{m-1} .$$

Similarly at $i = 1$, we add an extra line $i = 0$ such that $h_1 = h_0$ and then set

$$\chi_0 = \chi_2 .$$

The boundary condition at $j = 1$, i.e. at $r = 1$, is approximated in a different way. Here we use an interpolation between the three points (i,1), (i,2) and (i,3) and find that

$$-\chi_r\Big|_{r=1} = \frac{k_2 + 2k_1}{k_1(k_1 + k_2)}\chi_{i,1} - \frac{k_1 + k_2}{k_1 k_2}\chi_{i,2} + \frac{k_1}{k_2(k_1 + k_2)}\chi_{i,3} .$$

Since $\chi_r = -\cos\theta$ at $r = 1$, we have that

$$\chi_{i,1} = \frac{1}{k_2 + 2k_1}\left[\frac{1}{k_2}\left((k_1 + k_2)^2\chi_{i,2} - k_1^2\chi_{i,3}\right) + k_1(k_1 + k_2)\cos\theta\right]. \qquad \dots (3.5)$$

We can now sum the contribution (3.4) for each rectangle, and we see that $J[\chi]$ can be approximated by

$$J[\chi] = \bar{J} = \sum_{i=1}^{n}\sum_{j=1}^{m-1} J_{i,j} .$$

The values of $\chi_{i,j}$ which maximise this expression are given by the solutions to the equations

$$\frac{\partial\bar{J}}{\partial\chi_{i,j}} = 0 , \qquad \begin{array}{l} i = 1,\dots, n , \\ j = 2,\dots, m-1 . \end{array} \qquad (3.6)$$

These equations were solved by a Newton's method in the following way. For a given (i,j) we can write (3.6) in the form

$$g(\chi_{i,j}) = \sum_{s=1}^{4}\left[\alpha(A_s\chi_{i,j}^2 + B_s\chi_{i,j} + C_s)^{\alpha-1}(2A_s\chi_{i,j} + B_s) + D_s\right]H_s = 0$$

where A_s, B_s, C_s, D_s, H_s are functions of χ at the neighbouring points and of h_i, h_{i-1}, k_j, k_{j-1}. Let $\chi_{i,j}^{(q)}$ be the qth approximation to the solution.

Then an improved estimate is given by

$$\chi_{i,j}^{(q+1)} = \chi_{i,j}^{(q)} - \frac{g\left(\chi_{i,j}^{(q)}\right)}{g'\left(\chi_{i,j}^{(q)}\right)}$$

where $g'(z) = dg/dz$. This process is carried out for each point in turn with the calculated values being used as soon as they are available. In this way χ can be calculated to the desired degree of accuracy. When equations (3.6) are solved, (3.5) is then used to evaluate χ on the surface.

Calculations were also carried out for the regions $\pi/2 \leqslant \theta \leqslant \pi$, $1 \leqslant r \leqslant R$ and $\pi/2 \leqslant \theta \leqslant 3\pi/2$, $1 \leqslant r \leqslant R$. The boundary conditions at $\theta = \pi/2$ and $3\pi/2$ are for a body symmetric about $x = 0$ so that χ vanishes there.

The approach used by Greenspan and Jain[3] differs in some important aspects from the one described above. Given an interior point (i,j) they approximate the derivatives of χ by

$$\left(\frac{\partial \chi}{\partial r}\right)_{i,j} = \frac{\chi_{i,j+1} - \chi_{i,j}}{\Delta r} ,$$

$$\left(\frac{\partial \chi}{\partial \theta}\right)_{i,j} = \frac{\chi_{i+1,j} - \chi_{i,j}}{\Delta \theta} .$$

These expressions are then substituted into $J[\chi]$ to give an approximation $J_{i,j}$ for the rectangle (i,j), (i+1,j), (i+1,j+1), (i,j+1). A summation over all the points gives a global approximation J' to J. The boundary conditions are then used to obtain approximations to χ on the boundaries in terms of the neighbouring interior points, and these boundary values of χ are substituted into J'. A maximizing expression is found for J' by solving the equation

$$\frac{\partial J'}{\partial \chi_{i,j}} = 0$$

for all interior points.

The main differences between the two approaches are the treatment of the boundary conditions on the surface of the body and the extent to which the values of χ at the neighbouring points appear in the equation for $\chi_{i,j}$. Greenspan and Jain approximate the condition $\chi_r = -\cos\theta$ at $r = 1$ by writing

$$\chi_{i,0} = \Delta r \cos \theta_i + \chi_{i,1} \tag{3.7}$$

where (i,0) is on the surface, while we use a three points interpolation. Due to the way in which they approximate the derivatives of χ, their equation for $\chi_{i,j}$ depends only on χ at six of the eight neighbouring points, the points (i+1,j+1) and (i-1,j-1) being excluded. This is in contrast to the approach in this paper where χ at all the neighbouring points are used. It is difficult to know if these differences accounts for the fact that our solutions for flow past a circle are closer to other approximations near the stagnation points than those obtained by Greenspan and Jain.

4 CONVERGENCE

Two different convergence criteria were used. In the first one the iterative scheme was continued until the Mach number on the surface of the body changed by less than 1.0×10^{-5} during one iteration. This gives a solution of similar accuracy to that obtained by Sells[4]. It was found that this criterion was approximately fulfilled if the reduced potential did not change by more than the same amount during 100 iterations. Usually about 800 iterations were required, but this depended, of course, on the number of mesh points and on the value of the free stream Mach number. For the calculation of the flow past a 10% ellipse with $M_\infty = 0.8$ on a 17 × 21 grid on the region $1 \leqslant r \leqslant 21$, $0 \leqslant \theta \leqslant \pi$ 1000 iterations were required, and the computing time on an IBM 360 was about 20 minutes.

The rate of convergence was increased by using overrelaxation at the end of each iteration. Thus if $\bar{\chi}_{i,j}^{(q+1)}$ was obtained by solving equations (3.6) the new value of χ, $\chi_{i,j}^{(q+1)}$ was defined by

$$\chi_{i,j}^{(q+1)} = w\,\bar{\chi}_{i,j}^{(q+1)} + (1 - w)\,\chi_{i,j}^{(q)}$$

where w is a parameter greater than zero. If $w < 1$ we have underrelaxation and if $w > 1$ overrelaxation. It was found by trial and error that the best convergence for the ellipse was achieved with $w = 1.4$, but no extensive search for an optimal value was carried out. Even when the local Mach number was close to unity, it was not necessary to use underrelaxation in order to obtain convergence. This is in contrast to Sells[4].

A number of calculations were carried out for a 10% ellipse with a free stream Mach number of 0.8 on a quarter plane in order to test the importance of different grid sizes. Some results are given in Table 1, and they seem to indicate that if the number of points in either the radial or the angular direction is increased, the local Mach number on the surface converges.

Table 1

Local Mach numbers on the surface of a 10% ellipse with $M_\infty = 0.8$ for different grid sizes

	Grid				
θ	9 × 41	9 × 31	9 × 21	13 × 21	17 × 21
22.5^o	0.8617	0.8615	0.8606	0.8615	0.8621
45.0^o	0.9409	0.9413	0.9419	0.9447	0.9457
67.5^o	0.9683	0.9689	0.9702	0.9739	0.9752
90.0^o	0.9763	0.9769	0.9783	0.9825	0.9839

5 RESULTS

The main part of the calculations were carried out for non-lifting ellipses of different thickness ratios, but some results were also obtained for Karman-Trefftz profiles. In most of the calculations the computation region was $0 \leqslant \theta \leqslant \pi$, $1 \leqslant r \leqslant R$, and the solutions show that when the flow is subsonic the potentials for non-lifting ellipses are always symmetric about the line $\theta = \pi/2$. A few calculations were carried out for $\pi/2 \leqslant \theta \leqslant 3\pi/2$ in order to check the accuracy of the treatment of the boundary conditions at $\theta = 0$ and $\theta = \pi$. Since the results were identical with those obtained for $0 \leqslant \theta \leqslant \pi$ it was considered sufficient to only carry out the calculations for the upper half-plane. The results in this section for the different ellipses were obtained using the reduced incompressible solutions as boundary conditions at $r = R$, while for the Karman-Trefftz profile the boundary condition derived in the Appendix was used.

One of the simplest cases to consider is that of a circle for a free stream Mach number just below the critical value. Table 2 shows results obtained with a 21 by 21 grid and gives also for comparison similar results by the program developed by Sells[4]. Very good agreement is achieved.

Table 2

Local Mach numbers on the surface of a circle, $M_\infty = 0.39$

θ	Sells	Our results
0.0°	0.0	0.0
9.0°	0.1123	0.1123
18.0°	0.2246	0.2245
27.0°	0.3367	0.3366
36.0°	0.4483	0.4482
45.0°	0.5587	0.5585
54.0°	0.6665	0.6661
63.0°	0.7689	0.7682
72.0°	0.8604	0.8591
81.0°	0.9301	0.9276
90.0°	0.9582	0.9544

Table 3

Local Mach numbers on the surface of a 10% ellipse, $M_\infty = 0.8$

θ	Sells	Our results
0.00°	0.0	0.0
11.25°	0.7640	0.7250
22.50°	0.8900	0.8606
33.75°	0.9170	0.9132
45.00°	0.9398	0.9419
56.25°	0.9609	0.9590
67.50°	0.9756	0.9702
78.75°	0.9831	0.9762
90.00°	0.9855	0.9783

A more exacting test of the programme is to calculate solutions for thin bodies at high speeds. In Table 3 results are given for a 10% ellipse with a free stream Mach number of 0.8 and with the calculations done on a 17 by 21 grid. Again results obtained by Sells' programme are presented for comparison. The agreement away from the neighbourhood of the stagnation points is good, but near these points they differ by about 5%. It is not clear what the cause is of this difference. Since in both procedures conformal mappings are used to transform the ellipse into the unit circle, it would seem that the treatments of the region of high curvature are identical. It is possible that it is the different treatments of the boundary condition on the surface that are the cause. In Sells' programme the stream function is used so the boundary condition is the simple one of setting it equal to zero on the surface. In our programme, however, it is the normal derivative of the velocity potential which is prescribed. Different ways of treating this condition were tried out, but no improvement over the results presented here was achieved.

The values of the free stream Mach number were also increased until the numerical procedure ceased to converge. For a 10% ellipse with a 17 by 21 grid it converged for $M_\infty = 0.82$ but not for 0.83. In Table 4 the values of the local Mach number along the surface are given for different values of M_∞.

The flow past a Karman-Trefftz profile of 10% thickness and with a trailing-edge angle of 6^o was also calculated for different values of the free stream Mach number. The conformal transformation used is given by

$$\frac{z - mk}{z + mk} = \left(\frac{\sigma - 1}{\sigma - 1 + 2k}\right)^m$$

where the values $m = 1.9667$ and $k = 0.9375$ were used. Some selected values of the local Mach number are given in Table 5, see page 18.

6 DISCUSSION

It has been shown in this paper that the application of finite differences to the modified form of the Bateman-Dirichlet Principle can produce a satisfactory numerical solution for the plane flow past a non-lifting ellipse and a Karman-Trefftz profile. For a circle the results are much better than those obtained by Greenspan and Jain[3] and are in close agreement with the results of Sells[4]. The results for a thin non-lifting ellipse agree well with Sells' results except near the stagnation points where for a 10% ellipse with $M_\infty = 0.8$ the difference between the local Mach numbers is about 5%. No satisfactory explanation has been found, but it may be due to the different boundary conditions on the surface since Sells worked with a streamfunction which vanishes

Table 4

Local Mach numbers on the surface of a 10% ellipse for different values of M_∞

	M_∞ = 0.70	M_∞ = 0.80	M_∞ = 0.82
0.0°	0.00	0.00	0.00
11.25°	0.6569	0.7250	0.7351
22.50°	0.7587	0.8606	0.8768
33.75°	0.7888	0.9132	0.9357
45.00°	0.8019	0.9419	0.9696
56.25°	0.8081	0.9590	0.9940
67.50°	0.8115	0.9702	1.0089
78.75°	0.8131	0.9762	1.0242
90.00°	0.8136	0.9783	1.0281
101.25°	0.8131	0.9762	1.0213
112.50°	0.8115	0.9702	1.0119
123.75°	0.8081	0.9590	0.9922
135.00°	0.8019	0.9419	0.9704
146.25°	0.7888	0.9132	0.9353
157.50°	0.7587	0.8606	0.8771
168.75°	0.6569	0.7251	0.7349
180.00°	0.00	0.00	0.00

Table 5

Local Mach numbers on the surface of a 10% Karman-Trefftz profile with a trailing edge angle of 6° for different values of M_∞

	M_∞ = 0.60	M_∞ = 0.70	M_∞ = 0.75
0.0°	0.00	0.00	0.00
22.5°	0.5463	0.6291	0.6676
45.0°	0.5884	0.6833	0.7287
67.5°	0.6345	0.7452	0.8009
90.0°	0.6811	0.8122	0.8841
112.5°	0.7192	0.8715	0.9682
135.0°	0.7327	0.8886	0.9846
157.5°	0.6783	0.7936	0.8453
180.0°	0.00	0.00	0.00

at the surface while in the present work it is the normal derivative of the velocity potential that is set equal to zero.

It is planned to extend the present work to more realistic profiles. This will, in general, require a numerical evaluation of the transform modulus T as was done in Sells[4]. It will then be possible to obtain solutions for other thin bodies, besides the ellipse, and it will be interesting to see if these solutions will also be different from those calculated by Sells' programme near the stagnation points. Since the variational principle also holds for three-dimensional flows, it is also possible to change the programme so that the flow past axisymmetric bodies can be treated.

So far no attempt has been made to show the convergence of the numerical method used. However, if we only consider the reduced problem on the finite region $1 \leqslant r \leqslant R$ for given R it should be possible to show convergence by a modified form of the approach used in Rasmussen[7] where the convergence of the Rayleigh-Ritz method in plane subsonic flow is studied.

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Appendix

We now derive the alternative form of the boundary conditionson χ at $r = R$ for an ellipse. The basic idea is to find an asymptotic form of χ valid for large r which takes into account the shape of the body.

In a cartesian coordinate system the equation for the velocity potential ϕ can be written in the form

$$(c^2 - \phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (c^2 - \phi_y^2)\phi_{yy} = 0$$

where c is the local speed of sound. If a conformal transformation is used to map the ellipse onto the unit circle we obtain after some manipulation the following equation

$$U^2\left[\frac{1}{M_\infty^2} + \frac{\gamma - 1}{2}\left(1 - \frac{r^2\phi_r^2 + \phi_\theta^2}{r^2T^2U^2}\right)\right]\left(\frac{\phi_{\theta\theta}}{r^2} + \phi_{rr} + \frac{\phi_r}{r}\right)$$

$$- \frac{1}{r^4T^2}\phi_\theta^2\phi_{\theta\theta} - \frac{2}{r^2T^2}\phi_r\phi_\theta\phi_{r\theta} - \frac{\phi_r^2}{T^2}\left(\phi_{rr} + \frac{\phi_r}{r}\right)$$

$$+ \left(\phi_r^2 + \frac{\phi_\theta^2}{r^2}\right)\left[\frac{2\lambda^2 \sin^2\theta}{r^2T^4}\phi_\theta + \frac{\phi_r}{T^4}\left(1 - \frac{\lambda^4}{r^4}\right)\right] = 0 \qquad \text{(A-1)}$$

where $T^2 = \frac{1}{r^4}\left[(r^2 + \lambda^2) - 4\lambda^2 r^2 \cos^2\theta\right]$.

We now suppose that ϕ can be written in the form

$$\phi = U\left(r\cos\theta + \frac{1}{r}f(\theta)\right)$$

which is equivalent to supposing that

$$\chi = \frac{1}{r}f(\theta) .$$

When this is substituted into equation (A-1) and only the coefficients of r^{-3} are retained, an equation for $f(\theta)$ is obtained:

$$(1 - M^2 \sin^2 \theta)f'' - 4M_\infty^2 \sin\theta \cos\theta\, f' + \left[1 + (3 \sin^2\theta - 2)M_\infty^2\right]f = -2\lambda^2 M_\infty^2 \cos 3\theta .$$

The general solution of this equation is

$$f(\theta) = \frac{A \cos\theta + B \sin\theta - \lambda^2 M_\infty^2 \sin^2\theta \cos\theta}{1 - M_\infty^2 \sin^2\theta} \qquad \text{(A-2)}$$

where A and B are unknown constants. The potential ϕ must be symmetric about $\theta = 0$ and π and antisymmetric about $\theta = \pi/2$ and $3\pi/2$. Hence B must be equal to zero.

The constant A is found by comparing the calculated value of χ at $r = r_{n-1}$ to $r_{n-1}^{-1}\, f(\theta)$. Since χ is given by a set of values at $\theta_1, \theta_2, \ldots, \theta_n$, the method of least squares is used to give the best fit. The difference between the calculated χ and $r_{n-1}^{-1}\, f(\theta)$ at any i is

$$\frac{\cos\theta_i (A - \lambda^2 M_\infty^2 \sin^2\theta_i)}{r_{n-1}(1 - M_\infty^2 \sin^2\theta_i)} - \chi_{i,n-1} .$$

For the best fit we must minimize

$$D^2 = \sum_{i=1}^{m} \left[\frac{\cos\theta_i\ (A - \lambda^2 M_\infty^2 \sin^2\theta_i)}{r_{n-1}(1 - M_\infty^2 \sin^2\theta_i)} - \chi_{i,n-1}\right]^2 .$$

A necessary condition for A to give a minimum is that $dD^2/dA = 0$, and the solution to this equation is

$$A = \frac{r_{n-1} \sum_{i=1}^{m} \left[\frac{\chi_{i,n-1} \cos\theta_i}{1 - M_\infty^2 \sin^2\theta_i} + \frac{\lambda^2 M_\infty^2 \cos^2\theta_i \sin^2\theta_i}{r_{n-1}(1 - M_\infty^2 \sin^2\theta_i)^2}\right]}{\sum_{i=1}^{m} \frac{\cos^2\theta_i}{(1 - M_\infty^2 \sin^2\theta_i)^2}} .$$

This constant can then be evaluated at the end of each iteration and from (A-2) we see that χ at $r = R$ is given by

$$\chi = \frac{\cos\theta\ (A - \lambda^2 M_\infty^2 \sin^2\theta)}{R(1 - M_\infty^2 \sin^2\theta)} \quad . \tag{A-3}$$

For the flow past a 10% ellipse and a free stream Mach number of 0.8 the final value of A was 1.11450.

It is seen from the expression for χ *that for large values of* R this expression differs only little from the incompressible solution $\chi = R^{-1}\cos\theta$. However, if only 10 grid lines are used in radial direction, as in Sells[4], the corresponding value of R would be fairly small, and in this case it would be necessary to use (A-3) as boundary condition. The change in A between two consecutive iterations gives also an indication of the rate of convergence of the solution.

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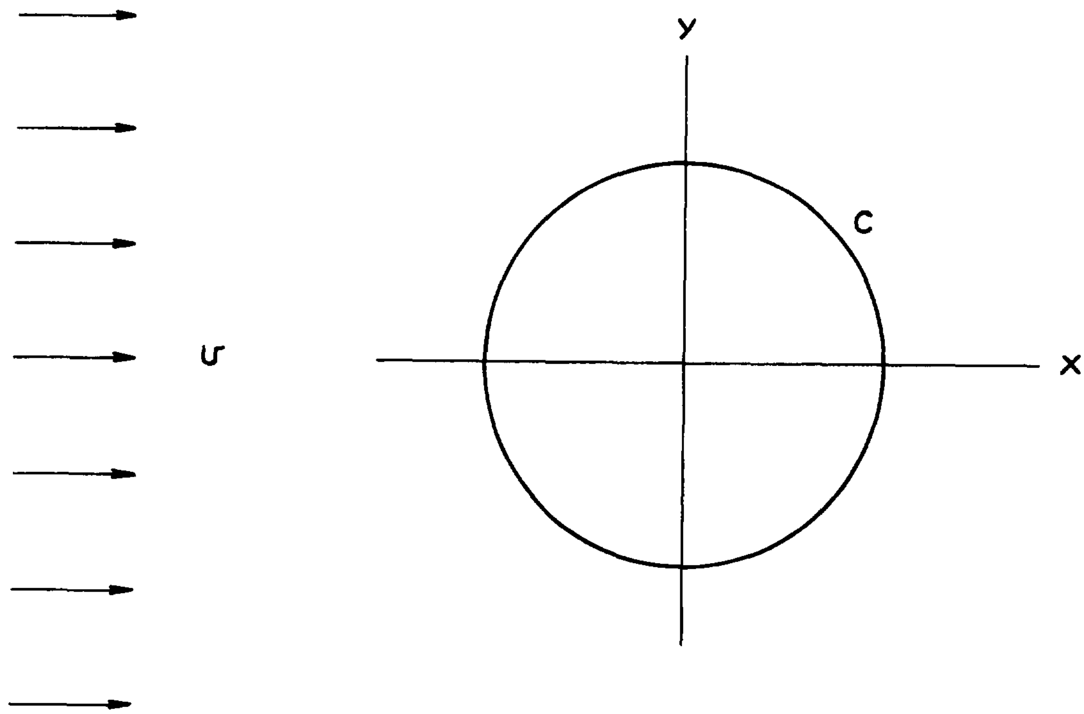

Fig.1 Sketch of flow field

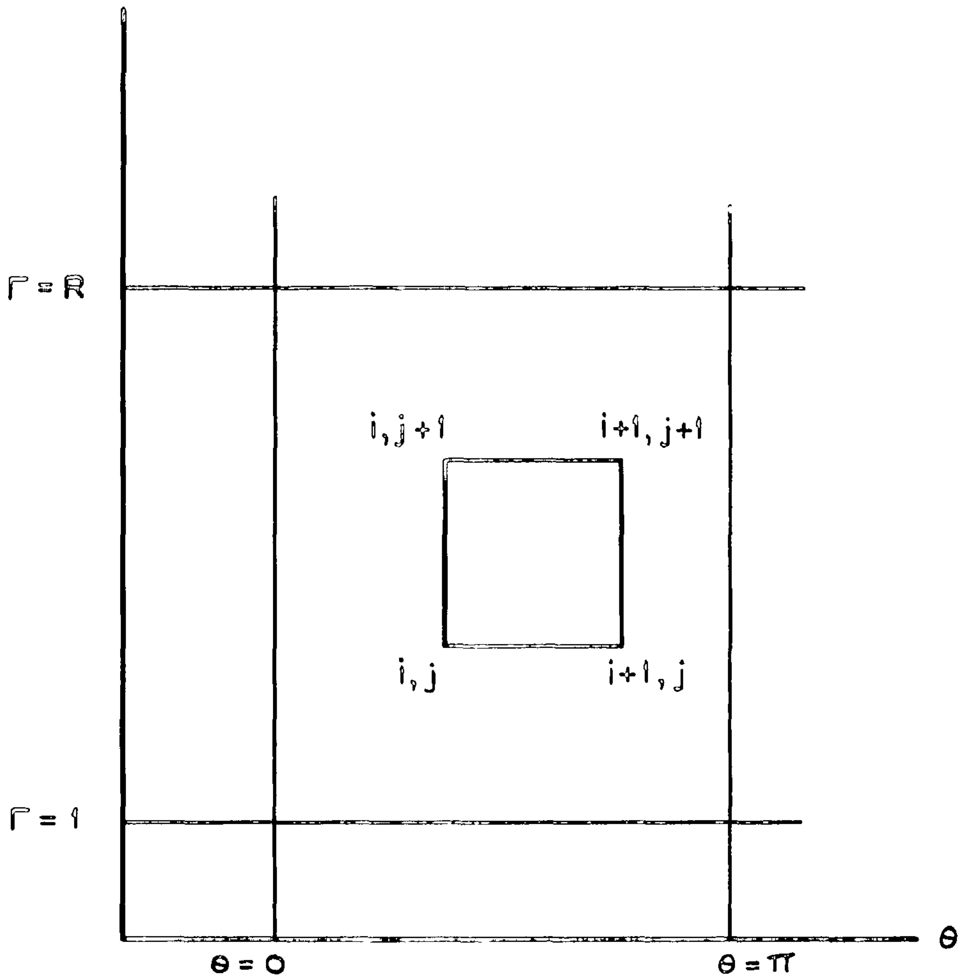

Fig.2 Computational grid

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