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The Pressure on the Surface of a Flat Elliptic Cone set Symmetrically in a Supersonic Stream

By

D. G. Hurley, B.A., B.Sc.,
of the Department of Supply, Australia

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7th November, 1951

Summary

The first order solution of the problem of the supersonic flow past a flat elliptic cone set symmetrically to the wind indicates that the pressure over the surface is constant if the body lies within the Mach cone of the apex. This result is incorrect near the leading edges of the cone and an improved solution is derived here by the introduction of line sources near the leading edges. Numerical results are given for three bodies.

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List of Symbols

a	-	speed of sound
b	-	semi-span of body
c	-	chord of body
c_1, c_2	-	constants
c_p	-	pressure coefficient
h, K	-	constants
l_x, l_y, l_z	-	direction cosines of the outward normal to the surface of the body
m	-	the tangent of the angle between the $+^{ve}$ x-axis and the line source
n	-	a constant: $n = \sqrt{M^2 - 1}$
M	-	Mach number
t	-	semi-thickness of body
u, v, w	-	velocity perturbations in the directions of the x, y and z axes
u_1, v_1, w_1	-	velocity components derived from ϕ_1
u_2, v_2, w_2	-	velocity components derived from ϕ_2
r, μ, ν	-	co-ordinate system - see reference 2
V	-	free stream velocity
x, y, z	-	system of cartesian co-ordinates - see Fig.1
α	-	apex semi-angle of body in horizontal plane of symmetry
γ	-	ratio of specific heats
ϕ	-	velocity perturbation potential
ϕ_1	-	see equation (6)
ϕ_2	-	see equation (14)
μ_1	-	a value of μ slightly larger than K. The surface $\mu = \mu_1$ defines the body on which the pressures are found.

1./

1. Introduction

The supersonic flow past a flat conical body of elliptic cross-section set symmetrically to the wind is considered in the case when the body lies entirely within the Mach cone at its vertex. This problem was considered in reference 1 and the result that the pressure is the same at all points on the surface of the body was obtained. This result cannot hold near the leading edge where, as is well known, large changes in pressure occur. The effect of these pressures on the drag of the body was allowed for by calculating the leading edge force but this procedure does not give the distribution of pressure near the leading edge of the body. The modification of the solution to yield this pressure distribution is the object of the present investigation. The shape of the body and the notation used are shown in Fig.1.

2. Statement of Problem

Consider the flow past the body as shown in Fig.1. When α, the apex semi-angle in the horizontal plane of symmetry, is small the velocity perturbations near the leading edge of the body are small. Hence it is reasonable to expect that the linearized equations of motion will give reasonably accurate values for the pressures at all points on the surface of the body provided α is not too large.

The linearized equation for the induced velocity potential ϕ is

$$(M^2 - 1)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \ldots (1)$$

where $M = V/a$ is the Mach number, V the free stream velocity and a the speed of sound ϕ is the induced velocity potential so that the three velocity components are given by

$$-V + \frac{\partial\phi}{\partial x}, \ \frac{\partial\phi}{\partial y} \ \text{and} \ \frac{\partial\phi}{\partial z} \ \text{respectively.}$$

The boundary conditions to be satisfied by ϕ are:-

(i) At the Mach cone

$$x^2 - (M^2 - 1)(y^2 + z^2) = 0, \ \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial z} = 0 \qquad \ldots (2)$$

(ii) At the surface of the body the resultant normal velocity is zero so that

$$l_x V = l_x \frac{\partial\phi}{\partial x} + l_y \frac{\partial\phi}{\partial y} + l_z \frac{\partial\phi}{\partial z} \qquad \ldots (3)$$

where l_x, l_y, l_z are the direction cosines of the outward normal at the surface of the body.

Having/

Having found ϕ and the velocity perturbations u, v and w, then according to strict linear theory the pressure coefficient, c_p, is given by

$$c_p = + \frac{2u}{V} . \qquad \dots (4)$$

However, it is found that near the leading edge c_p as given by (4) differs appreciably from the value given by the exact relation

$$c_p = \frac{2}{\gamma M^2}\left\{\left[1 + \frac{\gamma - 1}{2} M^2 \left(1 - \frac{(V-u)^2 + v^2 + w^2}{V^2}\right)\right]^{\frac{\gamma}{\gamma - 1}} - 1\right\} \dots (5)$$

Although it is somewhat inconsistent mathematically, relation (5) will be used to determine c_p.

3. Method of Solution

(i) 1st Approximation

In reference 1 it is shown that the problem under consideration is solved by the function

$$\phi = c_1 \phi_1 = c_1 x \mu \nu I(\mu) \qquad \dots (6)$$

(see refs. 1 and 2 and the next paragraph for the definitions of the symbols) if the exact boundary condition at the surface of the body, (3), is replaced by the approximate one

$$l_x V = \left(\frac{\partial \phi}{\partial z}\right)_{z = 0} . \qquad \dots (7)$$

Equation (7) is obtained from (3) by neglecting l_x and l_y in comparison with l_z (which is taken to be unity) and applying the condition at the plane $z = 0$ instead of at the surface of the body. These approximations are valid over the whole of the body except near the leading edge where they do not hold. It is therefore reasonable to take $c_1 \phi_1$ as the 1st approximation to ϕ and higher approximations will be obtained by adding functions that are significant only near the leading edge of the body.

The velocity components u_1, v_1 and w_1 corresponding to ϕ_1 are given by the following formulae:-

$$\left.\begin{aligned} u_1 &= - \frac{hk\, I(\mu)}{n} - \frac{\nu^2}{nhk\mu} \frac{(\mu^2 - h^2)^{\frac{1}{2}} (\mu^2 - k^2)^{\frac{1}{2}}}{\mu^2 - \nu^2} \\ v_1 &= - \frac{\nu(\nu^2 - h^2)^{\frac{1}{2}} (\mu^2 - k^2)^{\frac{1}{2}}}{h\, n\, (\mu^2 - \nu^2)} \\ w_1 &= - \frac{\nu(\mu^2 - h^2)^{\frac{1}{2}} (k^2 - \nu^2)^{\frac{1}{2}}}{k\, n\, (\mu^2 - \nu^2)} \end{aligned}\right\} \dots (8)$$

(ii)/

(ii) <u>The r, μ, ν Co-ordinate System</u>

The symbols r, μ, ν which occur in (6) denote a system of hyperboloido-conal co-ordinates (reference 2) which is related to the cartesian co-ordinates x, y, z of Fig.1 by the relations

$$x = -\frac{n r \mu \nu}{h k}, \quad y = \frac{r\sqrt{\mu^2 - h^2}\sqrt{\nu^2 - h^2}}{h n}, \quad z = \frac{r\sqrt{\mu^2 - k^2}\sqrt{k^2 - \nu^2}}{k n} \quad \ldots (9)$$

where

$$n^2 = M^2 - 1 = k^2 - h^2 \quad \ldots (10)$$

The μ = constant surfaces, obtained by eliminating r and ν from the relations (9), are given by

$$\frac{x^2}{n^2\mu^2} - \frac{y^2}{\mu^2 - h^2} - \frac{z^2}{\mu^2 - k^2} = 0. \quad \ldots (11)$$

Equation (11) represents a family of elliptic cones. As $\mu \to \infty$ the cones approach the Mach cone $x^2 - n^2(y^2 + z^2) = 0$, and as $\mu \to k$ they approach the two sided angular region in the (x, y) plane given by

$$\frac{x^2}{k^2} - y^2 > 0, \quad z = 0$$

<u>The surface $\mu = \mu_1$ ($\mu_1 = k + \delta k$ where δk is small) is therefore a flat elliptic cone lying close to the plate $\mu = K$ and is taken to define the body on which the pressure distribution is to be determined.</u>

The relations between μ_1, h, k and the quantities M, t/c, b/c of Fig.1 are

$$\left.\begin{aligned} \mu_1^2 &= \frac{k^2}{1 - n^2\left(\frac{t}{c}\right)^2} = \frac{h^2}{1 - n^2\left(\frac{b}{c}\right)^2} \\ \text{and} \quad k^2 - h^2 &= n^2 = M^2 - 1 \end{aligned}\right\} \quad \ldots (12)$$

(iii) <u>Higher Approximations</u>

Higher approximations to ϕ are obtained by taking

$$\phi = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + \ldots\ldots$$

and determining the constants c_n so that the boundary condition (3) is exactly satisfied at a finite number of points. Here ϕ_2, ϕ_3, etc. are the velocity potentials of supersonic line sources in the (x, y) plane

situated/

situated near the leading edge of the body (see Fig.2). The sources start at the origin and are semi-infinite in length. It is shown in Appendix I that the flow due to the sources is conical as required if the strength of each source varies linearly with distance from the origin, and also that it satisfies the boundary condition (2) at the Mach cone.

By a method of trial and error it was found that very good results (see §5) could be obtained by simply taking a pair of line sources each passing through one set of the foci of the elliptic sections of the body (see Fig.2) and determining the constants c_1 and c_2 so that the boundary condition (3) was exactly satisfied at the centre section and leading edge of the body. Further, it was found that adding extra sources and so satisfying (3) more exactly produced very little change in the resultant pressure distribution. Therefore, ϕ is taken in the form

$$\phi = c_1 \phi_1 + c_2 \phi_2 \qquad \ldots (13)$$

with (see Appendix I)

$$\phi_2 = \sqrt{x^2 - n^2(y^2 + z^2)} - \frac{x - m n^2 y}{\sqrt{1 - m^2 n^2}} \operatorname{arcosh} \left\{ \frac{x - m n^2 y}{n\left[(mx - y)^2 + z^2(1 - m^2 n^2)\right]^{\frac{1}{2}}} \right\} \ldots (14)$$

where artan m is the inclination of the line source to the $+^{ve}$ x-axis. The formula (13) applies to the half of the body for which $y > 0$ and in this region the effect of the source situated in the other half of the body is negligible.

In Appendix I it is shown that the velocity components u_2, v_2, w_2 derived from ϕ_2 are

$$\left.\begin{aligned}
u_2 &= \frac{-1}{\sqrt{1 - m^2 n^2}} \operatorname{arcosh} \theta + m(mx - y) \; f(x, y, z) \\
v_2 &= \frac{m n^2}{\sqrt{1 - m^2 n^2}} \operatorname{arcosh} \theta - (mx - y) \; f(x, y, z) \\
w_2 &= z(1 - m^2 n^2) \; f(x, y, z) \\
&\text{where} \\
\theta &= \frac{x - m n^2 y}{n\{(mx - y)^2 + z^2(1 - m^2 n^2)\}^{\frac{1}{2}}} \\
&\text{and} \\
f(x, y, z) &= \frac{\sqrt{x^2 - n^2(y^2 + z^2)}}{\left[(mx - y)^2 + z^2(1 - m^2 n^2)\right]}
\end{aligned}\right\} \ldots (15)$$

4./

4. The Application of the Method to Examples

The method has been applied to 3 bodies each of apex semi-angle 30° and of thicknesses 5%, 10% and 15% in a stream of Mach number $\sqrt{2}$. The application to the 15% thick body will be described to illustrate the method.

We have $M^2 = 2$ and, from Fig. 1, $b/c = \tan 30°$ and $t/c = 0.75$. These values substituted into equation (12) yield $h = 1.4263$, $k = 1.7419$ and $\mu_1 = 1.7469$.

The values of (u_1, v_1, w_1) and $(y/x, z/x)$ are now calculated from formulae (8) and (9) for a range of values of ν, i.e., various points round the section of the body. Since the body has two planes of symmetry only points in one quadrant of its surface need be considered. The direction cosines (l_x, l_y, l_z) of the normal to the surface at these points are obtained from

$$\left.\begin{aligned} l_x &= \frac{1}{\sqrt{D}}, \quad l_y = \frac{\dfrac{y}{x}}{\left(\dfrac{b}{c}\right)^2 \sqrt{D}}, \quad l_z = \frac{\dfrac{z}{x}}{\left(\dfrac{t}{c}\right)^2 \sqrt{D}} \\ \text{where} \quad D &= 1 + \frac{\left(\dfrac{y}{x}\right)^2}{\left(\dfrac{b}{c}\right)^4} + \frac{\left(\dfrac{z}{x}\right)^2}{\left(\dfrac{t}{c}\right)^4} \end{aligned}\right\} \quad \ldots (16)$$

The constant c_1 is now determined by satisfying the boundary condition (3) at the point $\frac{y}{x} = 0$ $\frac{z}{x} = 0.075$. It is permissible to do this before u_2, v_2 and w_2 are found as these are negligible at this point.

The positions of the foci of the elliptic sections are now calculated and this gives a value of m for substitution into (15) from which u_2, v_2 and w_2 are found. The value of c_2 is determined by satisfying the boundary condition (3) at the leading edge of the body, the velocity components u, v and w now follow and c_p is calculated from formula (5).

5. Results and Discussion

(i) The Boundary Condition at the Surface of the Body

Figs. 3, 4 and 5 show, plotted against the vertical ordinate z/x, the L.H.S. of equation (3) and also the R.H.S. for the 1st and 2nd approximate solutions. It is seen that in each case the 1st approximation

satisfies/

satisfies the boundary condition (3) well except for z/x small (i.e., near the leading edge) whereas the 2nd approximation satisfies it well over the whole surface of the body.

(ii) The Pressure Distributions

Fig.6 shows the variation of the pressure coefficient, c_p , around the surface for each of the three bodies. The pressure coefficient at any point is represented by a line normal to the surface of length proportional to c_p .

It is of interest to compare the values of c_p at the leading edge with the value that is obtained by resolving V along and perpendicular to the leading edge and assuming the perpendicular component to vanish at the leading edge. Using equation (5) this latter value is $c_p = 0.283$ which is smaller than the values shown in the figures.

The values of c_p for the three bodies are given in Table I.

(iii) The Drag

The drag coefficient based on frontal area of the 10% thick body was obtained by integration of the pressure as $C_D = 0.0991$. This value compares well with the value 0.0992 given by strict linear theory, (i.e., by calculating the drag taking the surface pressure to be constant and adding to the result a leading edge force correction).

6. Conclusions

A method has been described for calculating the pressures on the surface of a flat conical body of elliptic cross-section set symmetrically in a supersonic stream, for the case when the body lies entirely within the Mach cone at its vertex. It is likely that the effect of incidence could be dealt with by a similar method.

7. Acknowledgement

The author wishes to thank Mr. H. B. Squire for the helpful suggestions he made throughout the work.

References

No.	Author(s)	Title, etc.
1	H. B. Squire	An Example in Wing Theory at Supersonic Speeds. R. & M.2549. February, 1947.
2	A. Robinson	Aerofoil Theory of a Flat Delta Wing at Supersonic Speeds. R. & M.2548. September, 1946.

TABLE I/

TABLE I

Values of c_p

$M = \sqrt{2}$; apex semi-angle = $30°$

$\frac{2t}{c} = 0.05$		$\frac{2t}{c} = 0.10$		$\frac{2t}{c} = 0.15$	
z/x	c_p	z/x	c_p	z/x	c_p
0.02500	0.0322	0.05	0.0622	0.0750	0.0901
0.02048	0.0326	0.0462	0.0628	0.0632	0.0935
0.01473	0.0344	0.0408	0.0642	0.0460	0.1057
0.01137	0.0370	0.0353	0.0665	0.0362	0.1222
0.00702	0.0480	0.0293	0.0709	0.0239	0.1717
0.00584	0.0558	0.0226	0.0809	0.0171	0.2291
0.00438	0.0747	0.0138	0.1204	0.0126	0.2804
0.00344	0.0991	0.0126	0.1312	0.0096	0.3094
0.00273	0.1315	0.0114	0.1458	0.0072	0.3220
0.00213	0.1746	0.0100	0.1667	0.0050	0.3286
0.00176	0.2104	0.0084	0.1980	0.0035	0.3353
0.00128	0.2592	0.0064	0.2471	0.00065	0.3466
0.000443	0.2970	0.0034	0.3074	0	0.3468
0	0.3100	0.0025	0.3131		
		0.00055	0.3300		
		0	0.3320		

APPENDIX I/

APPENDIX I

Formulae for Supersonic Line Sources

Consider the semi-infinite supersonic line source situated as shown in Fig.7, the strength of the source at a point $(t, mt, 0)$ on it being t.

The velocity potential $d\phi$ at any point (x, y, z) due to a length $\sqrt{1 + m^2}\,dt$ of the source at $(t, mt, 0)$ is then

$$d\phi = \frac{\sqrt{1 + m^2}\; t\; dt}{[(x - t)^2 - n^2\{(y - mt)^2 + z^2\}]^{\frac{1}{2}}}$$

for points inside the Mach cone of the point $(t, mt, 0)$, and $d\phi = 0$ for points outside this Mach cone.

It follows that the velocity potential due to the whole line source is

$$\phi(x, y, z) = \int_0^{t_1} \frac{\sqrt{1 + m^2}\; t\; dt}{[(x - t)^2 - n^2\{(y - mt)^2 + z^2\}]^{\frac{1}{2}}} \qquad \ldots (1)$$

for points inside the Mach cone of the origin. Here t_1, the upper limit of integration, is given by

$$(x - t_1)^2 = n^2\{z^2 + (y - m\,t_1)^2\}.$$

Evaluation of the integral in (1) gives the result

$$\phi = D\left\{\sqrt{x^2 - n^2(y^2 + z^2)} - \frac{x - mn^2 y}{\sqrt{1 - m^2n^2}}\,\mathrm{arcosh}\left\{\frac{x - m\,n^2\,y}{n\{(mx - y)^2 + z^2(1 - m^2n^2)\}}\right\}\right\} \ldots (2)$$

where D is a constant the value of which does not concern us.

The velocity components derived by differentiating (2) are

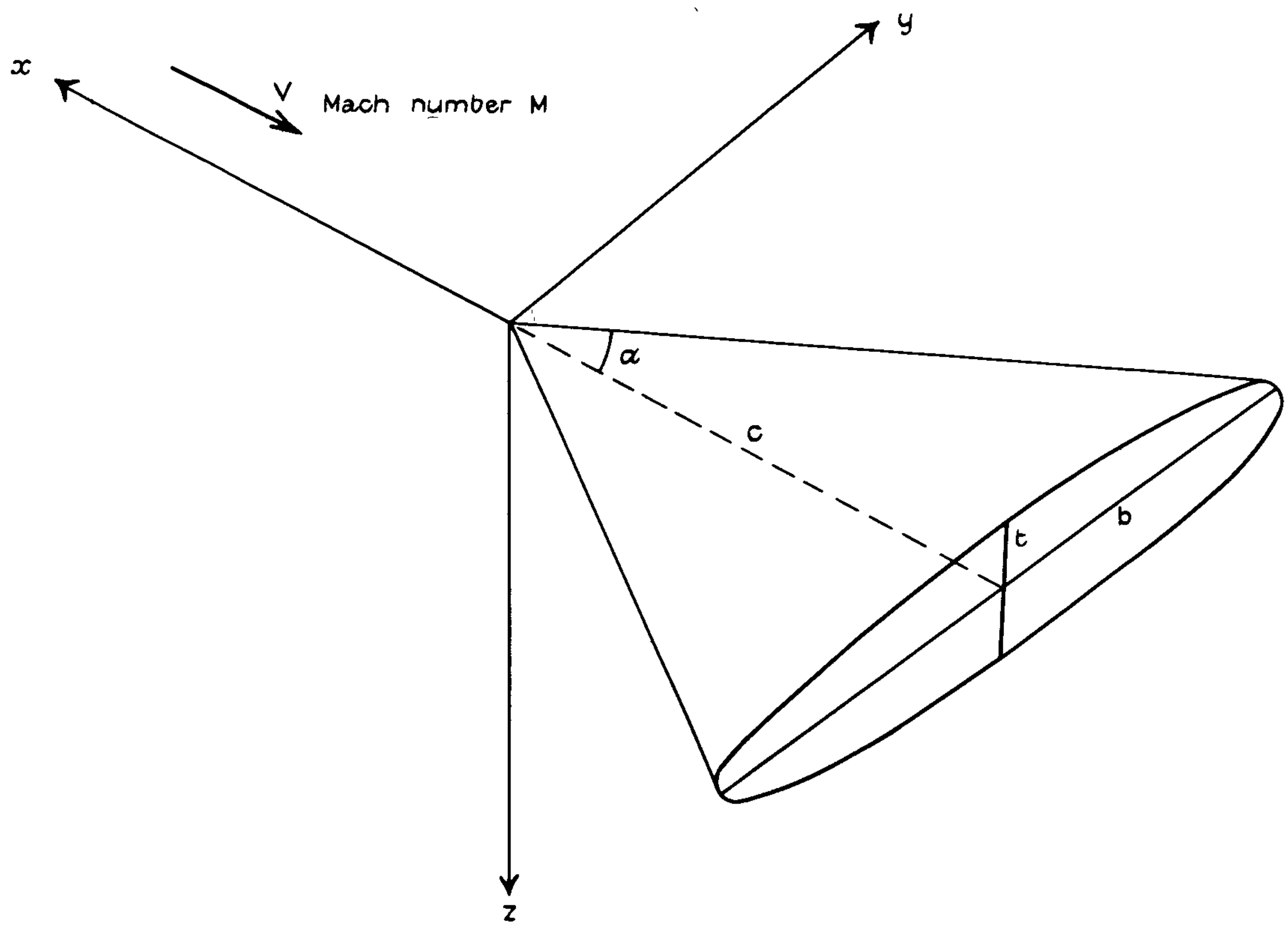
$$
\left.
\begin{aligned}
\frac{\partial\phi}{\partial x} &= D\left\{\frac{-1}{\sqrt{1 - m^2n^2}} \operatorname{arcosh}\theta + m(mx - y)\ f(x, y, z)\right\} \\
\frac{\partial\phi}{\partial y} &= D\left\{\frac{m\,n^2}{\sqrt{1 - m^2n^2}} \operatorname{arcosh}\theta - (mx - y)\ f(x, y, z)\right\} \\
\frac{\partial\phi}{\partial z} &= D\,z(1 - m^2n^2)\ f(x, y, z) \\
\text{where} & \\
\theta &= \frac{x - m\,n^2\,y}{n[(mx - y)^2 + z^2(1 - m^2n^2)]^{\frac{1}{2}}} \\
\text{and} & \\
f(x, y, z) &= \frac{\sqrt{n^2 - n^2(y^2 + z^2)}}{[(mx - y)^2 + z^2(1 - m^2n^2)]}\,.
\end{aligned}
\right\}
\quad \ldots (3)
$$

Each of the velocity components in (3) is of zero order in (x, y, z) and so the values are independent of $r = \sqrt{x^2 + y^2 + z^2}$, i.e., the flow is conical as required.

Also, as the Mach cone of the origin is approached $f(x, y, z) \to 0$ and $\theta \to 1$, so that all the velocity components vanish there as required.

JW

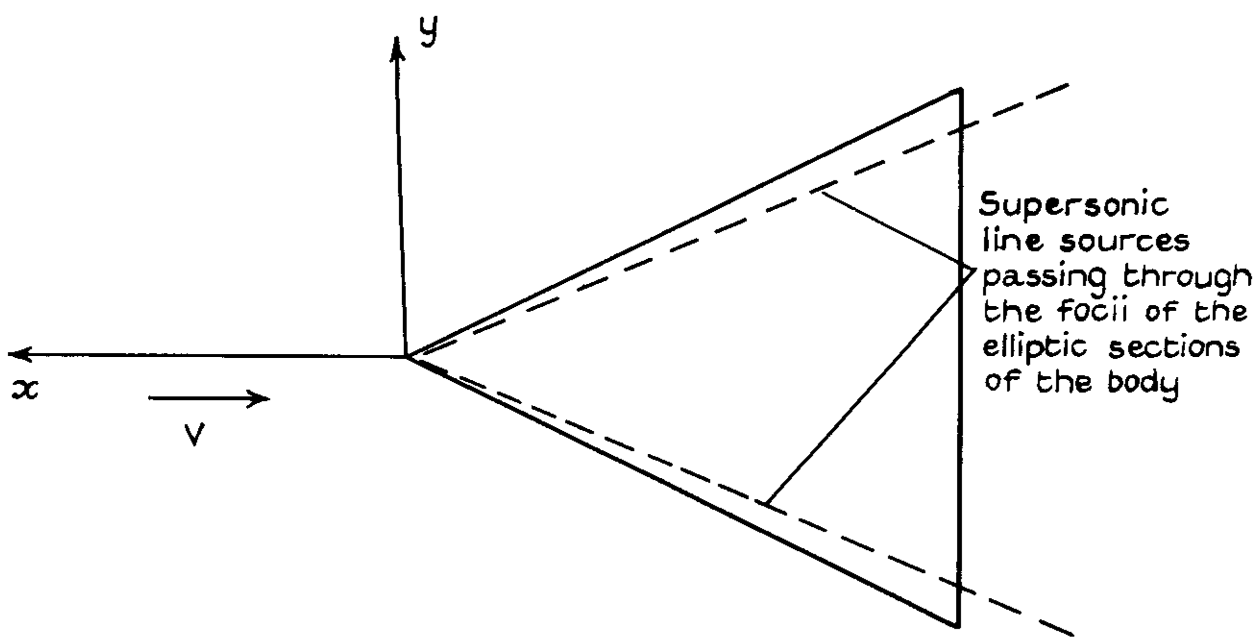
FIG. 1.



Shape of body and notation.

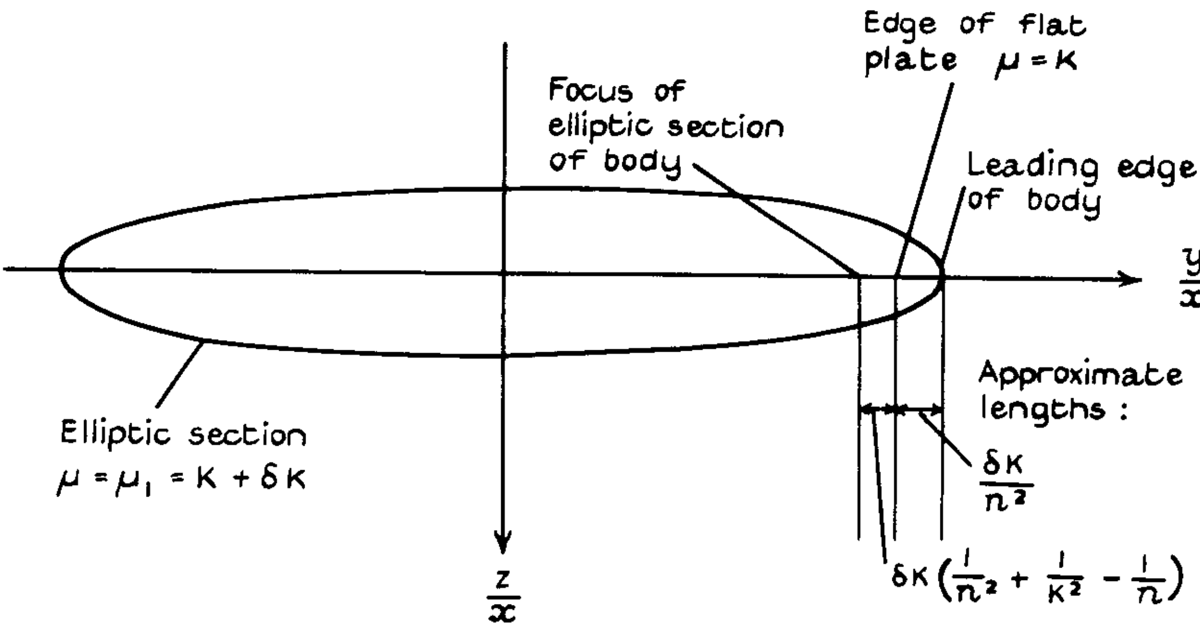
FIGS. 2a & 2b.

FIG 2a.



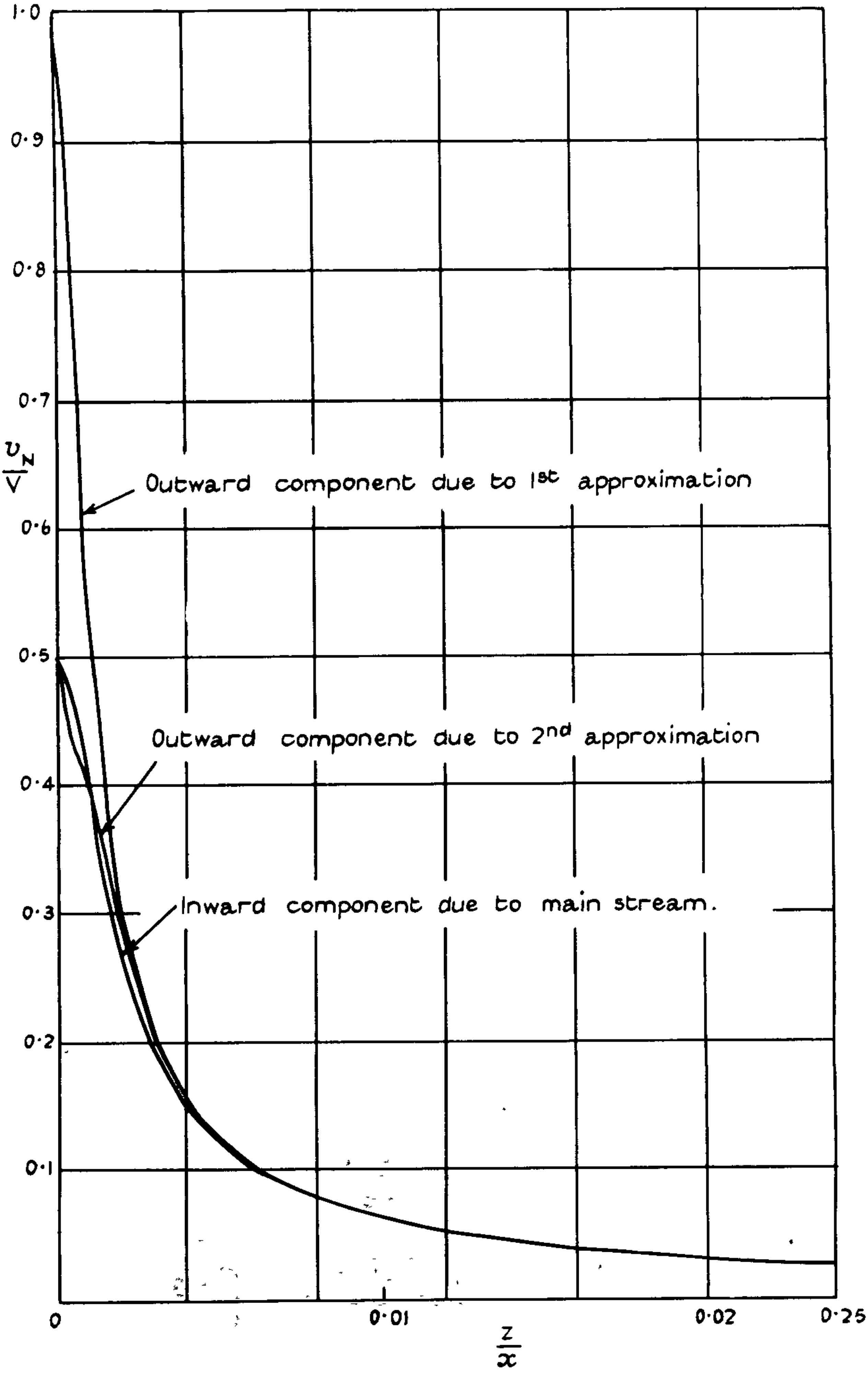
Plan view of body showing positions of supersonic line sources.

FIG. 2b.



Section of body in y, z planes.

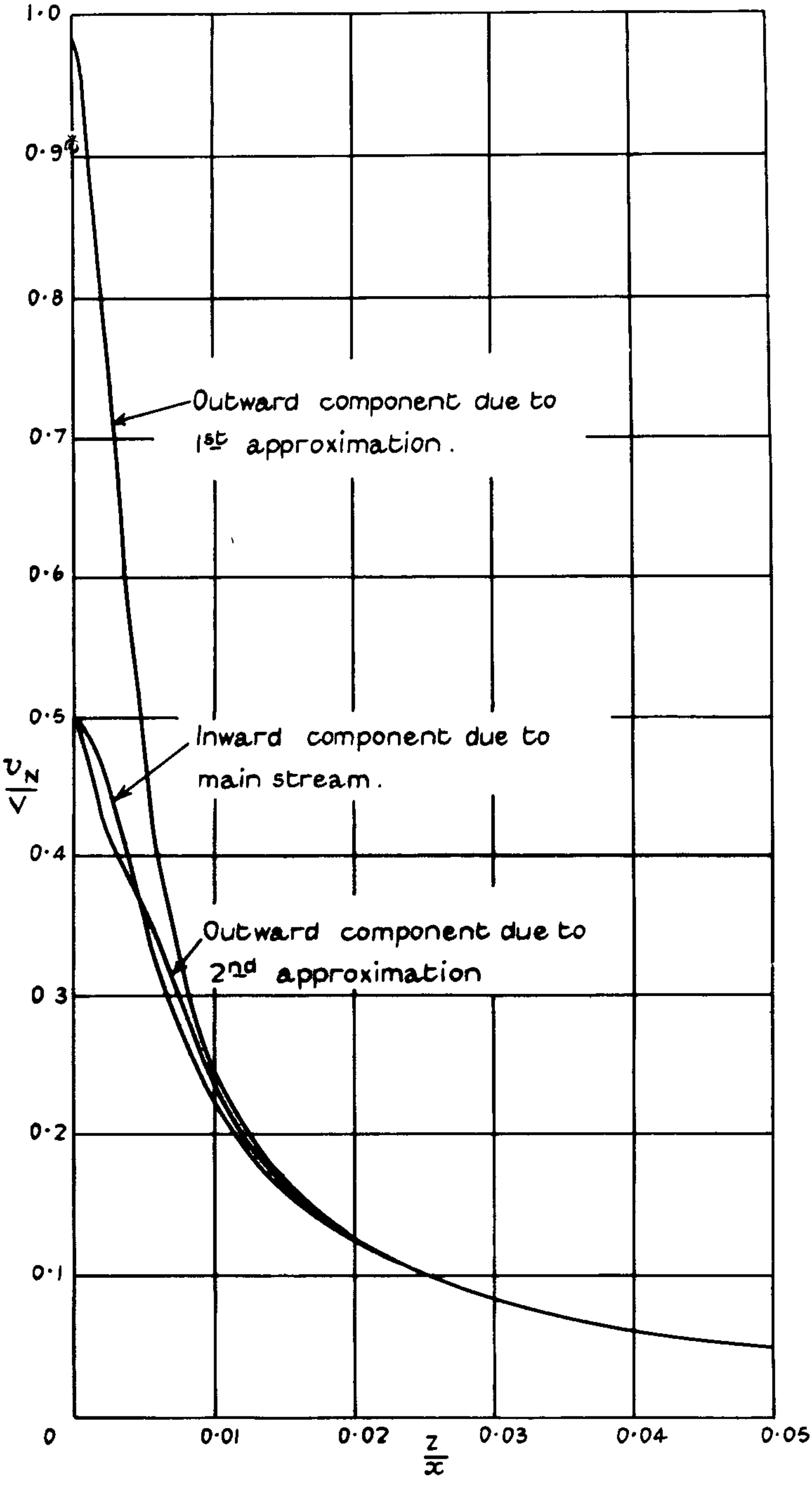
FIG. 3.



The Normal Velocity, v_N, across the Surface of the Body.

Body thickness 0·05
Semi-apex angle 30°
Free stream Mach No $\sqrt{2}$

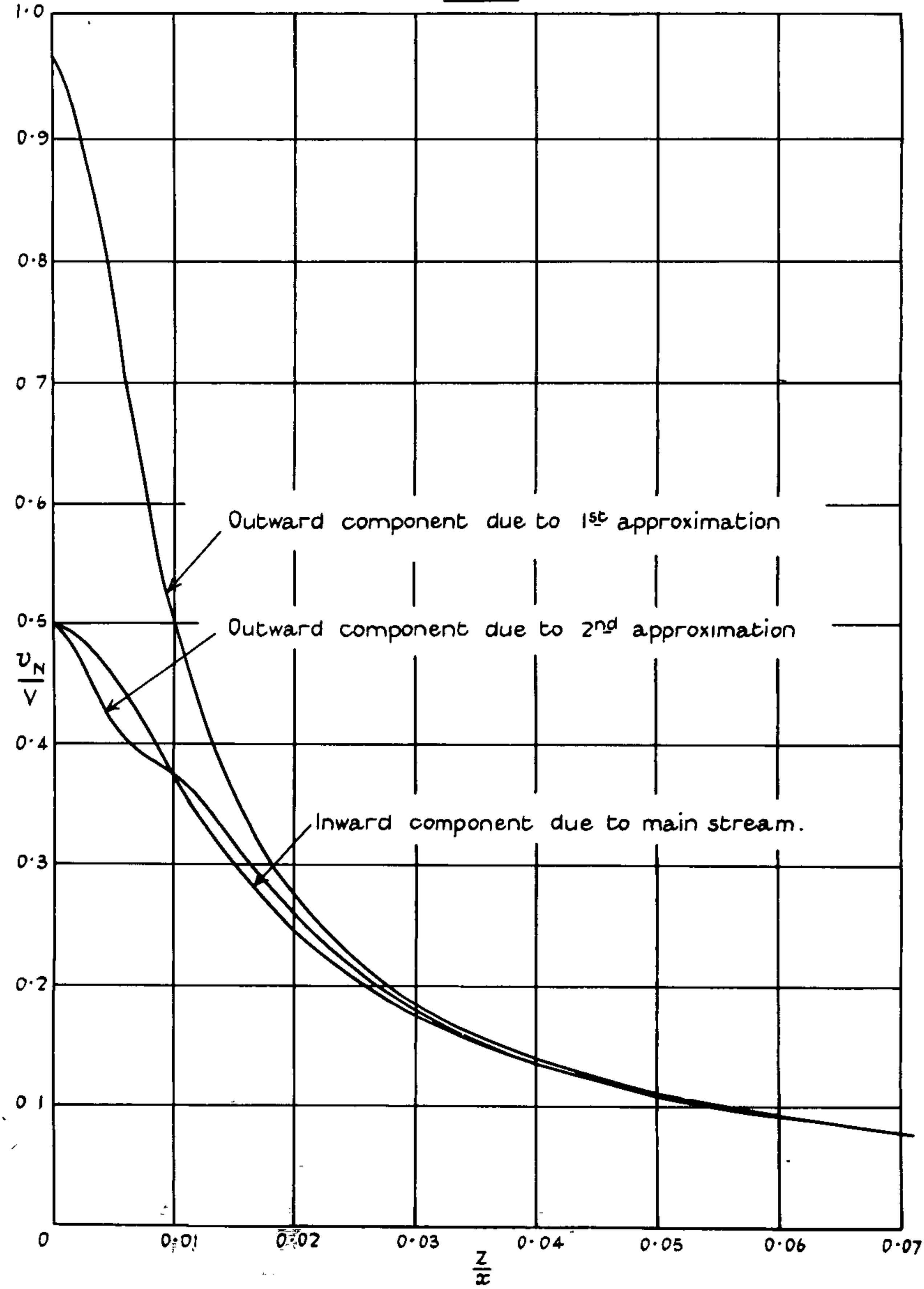
FIG. 4.



The Normal Velocity, v_N, across the Surface of the Body.

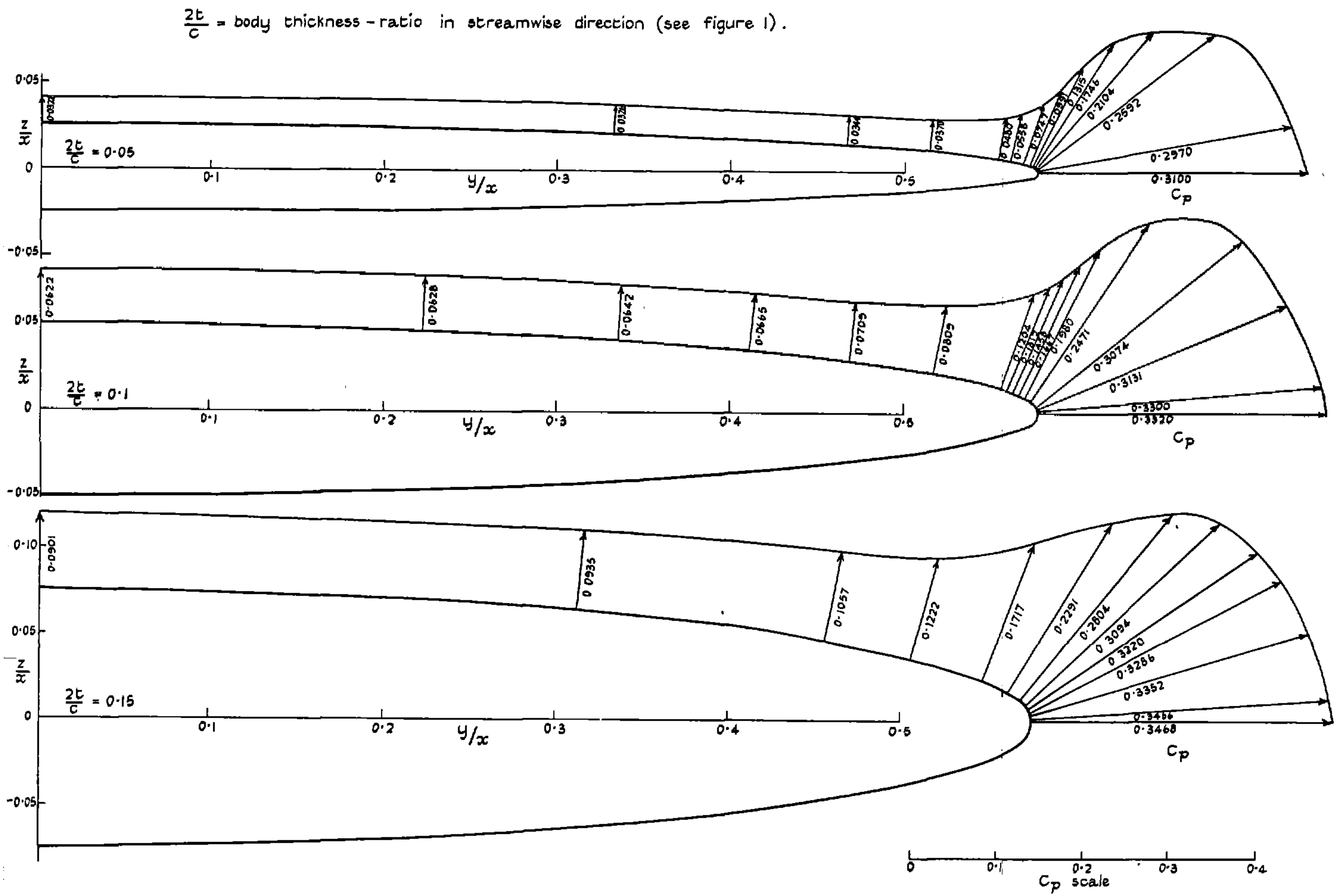
Body thickness 0·10
Semi-apex angle 30°
Free stream Mach No $\sqrt{2}$

FIG. 5.



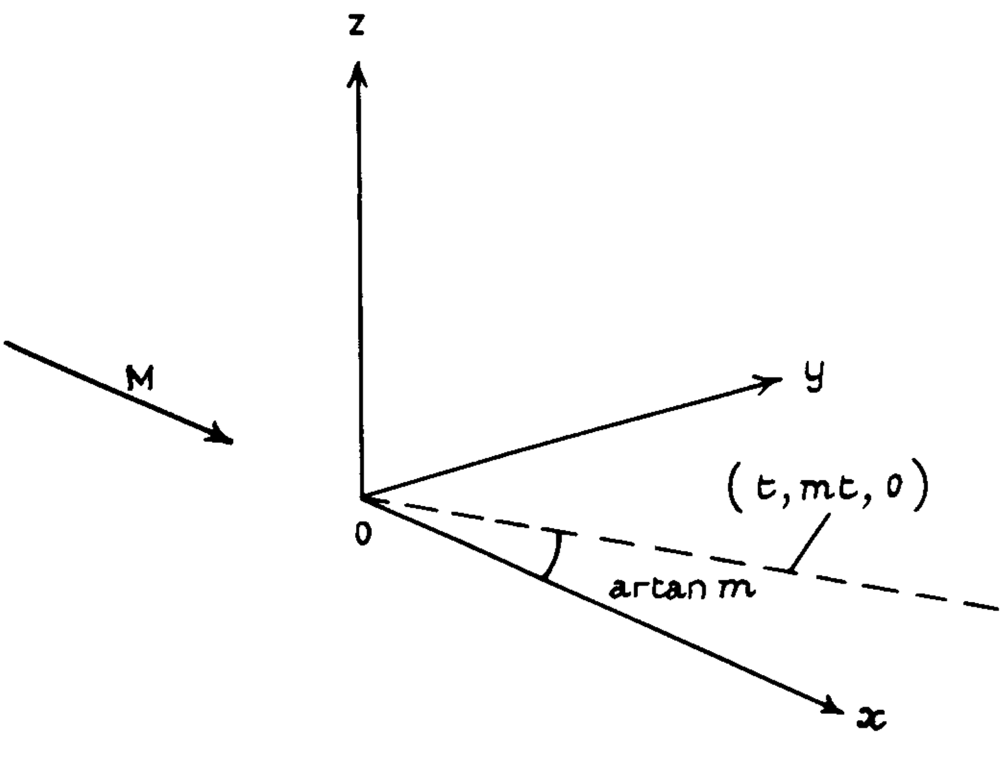
The Normal Velocity, v_N, across the Surface of the Body

Body thickness 0·15
Semi-apex angle 30°
Free stream Mach No $\sqrt{2}$



The Pressures on bodies of semi-apex angle 30° and different thicknesses in a stream of Mach number $\sqrt{2}$.

FIG. 7.



Notation for line source formulae.

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