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A Method for the Prediction of
the Probabilities of Aircraft
Fatigue Failures within
a Fleet of Known Size

by

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A METHOD FOR THE PREDICTION OF THE PROBABILITIES OF AIRCRAFT
FATIGUE FAILURES WITHIN A FLEET OF KNOWN SIZE

by

A. M. Stagg

SUMMARY

An algebraic solution is given for the problem of ascertaining the probability of finding any number of fatigue cracks in a fleet of aircraft at any specific point in time. By a simple approximation concerning the fatigue damage incurred by the members of the fleet the solution is simplified into a working method for the determination of the aforesaid probability.

With two parameters arbitrarily chosen, results are given to illustrate the nature of the solution.

* Replaces R.A.E. Technical Report 68181 - A.R.C. 30 817

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1 INTRODUCTION

In recent years the basic philosophy of the design of aircraft structures has undergone a modification. Structures which consist mainly of 'safe-life' parts, whose retirement from service is based on the probability of occurrence of a catastrophic crack which must not be allowed to occur, are being replaced by structures which consist mainly of 'fail-safe' parts in which the fatigue cracks that occur can be detected before the static strength falls to an unacceptable level. The full-scale fatigue test is at least as important for 'fail-safe' structures as for 'safe-life' structures in that it must demonstrate not only that all the cracks that are likely to occur in service are detectable, but it must also indicate the position and time at which the cracks occur and the speed at which they propagate.

The correlation of an aircraft in service with the full-scale fatigue test is achieved by the carriage in the service aircraft of a fatigue meter, the readings of which in conjunction with the fatigue meter formula give the fatigue index accumulated by that aircraft. Similarly the occurrence of cracks on the full-scale fatigue test can be related to the fatigue index scale, for it is from the full-scale fatigue test that the fatigue meter formula is derived. Thus a one-to-one correlation is established between service and test, which enables the maintenance engineers to decide when to start crack inspections at the various vulnerable stations.

This paper, being an extension to Marjorie Owen's paper¹, indicates a method for the calculation of the probabilities of occurrence of cracks in a fleet of aircraft at any time, taking into account both the individual aircraft fatigue indices and also the uncertainty of the estimate from the full-scale test of the crack distribution parameters. A knowledge of these probabilities should make possible a more economic and efficient system of stocking spares which in turn could lead to improved fleet servicing.

2 SCATTER IN FATIGUE

With the materials in use at present, there is considerable scatter in the fatigue strength of apparently identical specimens tested under identical loading conditions. A great many results indicate that the distribution of the logarithm of fatigue life is approximately normal; the evidence is less conclusive towards the asymptotes of the distribution. The log-normal distribution is thought to provide the most accurate and the

simplest definition of the distribution of scatter in fatigue life and for this reason the work hereafter is based on the assumption of the normality of the distribution of the logarithm of fatigue life.

As a result of the scatter in fatigue performance, if one or more specimens are tested the mean endurance achieved is only an estimate of the mean endurance of the whole population. Hereafter in this Report when the 'life' of a specimen is mentioned, the life meant is the time to the occurrence of that one particular crack in that specimen which is under consideration.

3 FATIGUE FACTORS

When the probability of failure of a specimen after a certain number of cycles is considered, allowance has to be made for the uncertainty of the estimate, from previous test results, of the population fatigue life. Bullen² considers three distinct methods of allowing for this uncertainty according to the state of knowledge of the distribution considered. Bullen's three cases are as follows:-

- (i) There is no prior knowledge of the mean or standard deviation of the distribution.
- (ii) There is no prior knowledge of the mean but the coefficient of variation is known.
- (iii) There is no prior knowledge of the mean but the standard deviation is known.

The factors on fatigue life quoted in Av.P.970 for application to test lives of structures are obtained from case (ii) from Bullen's paper, where a value of the coefficient of variation (V) for typical types of structures is assumed known.

It must be emphasized at this point that Bullen's paper and the work that follows apply only to a normal distribution. It was stated above that the work in this Report was based on the assumption of the normality of the distribution of the logarithm of fatigue life (N). If the fatigue strength x of a crack is defined to be the logarithm of the fatigue life of that crack, then

$$x = \log N$$

and on the assumption made x must be normally distributed and so can be used as the property under consideration in case (ii) of Bullen's paper.

Then if n samples of a specimen have been tested to give an estimated mean strength of \bar{x} and if x is the undetermined strength of a randomly selected single specimen going into service, Bullen's paper states that the statistic Z where

$$Z = \frac{\left\{ 1 - \frac{x}{\bar{x}} \right\}}{\sqrt{\left\{ 1 + \frac{x^2}{n\bar{x}^2} \right\}}^{\frac{1}{2}}}, \quad (1)$$

is distributed normally about zero mean with unit standard deviation.

The fatigue damage index accumulated by an aircraft is a measure on an arbitrarily chosen linear scale of the fatigue damage to which that aircraft has been subjected and is calculated by the substitution of the fatigue meter readings into the fatigue meter formula. It provides an easy basis for assessing the damage suffered by an aircraft in service and for matching the fatigue circumstances, at any time, of such an aircraft, as measured by the fatigue index, with the known circumstances of a previously tested aircraft at some period during that test.

Consider the case of a crack which occurred under testing at fatigue indices of t_1, t_2, t_3 etc. on various tests. Now the fatigue index is a linear scale, whilst the fatigue life is log-normally distributed, and so the fatigue index (\bar{T}) corresponding to the mean occurrence of this crack is given by

$$\begin{aligned} \bar{T} &= \text{antilog} \left\{ \frac{1}{n} (\log t_1 + \log t_2 + \log t_3 + \text{etc.}) \right\} \\ &= \text{antilog} \left\{ \frac{1}{n} \log (t_1 t_2 t_3 \dots) \right\} \end{aligned}$$

It is customary practice in determining a fatigue life to apply a factor A to the mean time of occurrence of a crack to allow for the scatter in fatigue performance. Hence the fatigue index (L_T) at which the probability of occurrence of this crack is $1/n$ or, in other words, at a time corresponding to a point 3 standard deviations below the mean of the density

distribution of occurrences of that crack, and thus the fatigue index (L_I) at which inspections for this crack should start, is given by

$$A L_I = \bar{T} \quad (2)$$

4 SERVICE LIFE DISTRIBUTION

Previous experience gained from failures in service of other aircraft suggests that the environment encountered in service is detrimental to the fatigue life of a specimen and reduces the mean life of the distribution of lives of specimens in service from the test mean life \bar{T} to a service mean life \bar{S} . Although no real quantitative evidence exists, it is considered that a reasonable but conservative relationship between \bar{T} and \bar{S} can be taken to be

$$\bar{S} = \frac{\bar{T}}{2} \quad (3)$$

Similarly it is felt that whilst the mean is reduced, the scatter is also reduced such that the fatigue index corresponding to a probability of failure of 1 in 740 remains the same for both sets of conditions, Fig.1. The scatter in the fatigue life distribution is thus altered, but care must be taken to note that the standard deviation is not halved as is the mean for the factor of two is applied to the antilogarithm of the logarithmic mean and not directly to the logarithmic mean, whilst we are concerned with logarithmically normal distributions.

5 INDIVIDUAL PROBABILITY OF FAILURE

Inspections for possible cracks should be started when the probability of that crack having already occurred is 1 in 740. L_I has already been defined as the fatigue index at which inspections should be started and \bar{S} as the fatigue index corresponding to the mean occurrence of the crack under consideration. Thus using equation (1) the coefficient of variation can be determined:-

$$v = \frac{1 - \frac{\log(L_I)}{\log(\bar{S})}}{3 \left\{ 1 + \frac{1}{n} \left(\frac{\log L_I}{\log \bar{S}} \right)^2 \right\}^{\frac{1}{2}}} \quad (4)$$

Reverting to equation (1) a plot can be drawn of the probability of cracking of any given aircraft when a fatigue index of y has been reached. Figs.2-5 present typical examples of such a plot on normal probability scale for values of V assumed at 0.05, 0.10, 0.50 and 1.00 in terms of \log_{10} and various values of n , the number of sample specimens that had been tested to provide the estimated mean \bar{S} . It is seen in this figure that as n increases the distribution becomes nearer and nearer the limiting case when $n = \infty$. This limit is the parent normal distribution, for then the estimated mean becomes, by definition, the population mean and so the plot for $n = \infty$ is a straight line.

6 COMBINED PROBABILITY OF FAILURE

It must be emphasized that Figs.2-5 are graphs of the probability of cracking of any particular aircraft in a fleet by the time it has reached a certain fatigue damage index. If (Fig.6) aircraft 1 is considered, there is a probability of P_1 of there being a crack present when it has reached b_1 fatigue index, a probability of P_2 of a crack when it has reached b_2 fatigue index, and so on. The same figures apply exactly to all the individual aircraft of the one fleet. The probability of cracking of r aircraft in a fleet of m aircraft at some date or instant of time will now be considered.

The fatigue damage index, or in other terms the damage accumulated, by the aircraft in a fleet will vary with time, increasing according to some function of time t . The damage D_i accumulated by the general aircraft (1) is thus given by

$$D_i = f_i(t) \quad (5)$$

The probability of cracking (P) of any individual aircraft is given as a function of the damage by a plot such as Fig.2; the plot being the same for all aircraft of one fleet say

$$P = g(D) \quad (6)$$

If at time t the probability of cracking of aircraft (i) in a fleet of m aircraft is P_i , the probability of non-cracking of aircraft (i) is given by q_i where

$$q_i = 1 - P_i \quad (7)$$

The probability $P_r(t)$ of exactly r aircraft in the fleet having cracks present by time t is the coefficient of t^r in the binomial expansion

$$(P_1 t + q_1) (P_2 t + q_2) \dots (P_i t + q_i) \dots (P_m t + q_m) \quad (8)$$

Thus

$$P_r(t) = W \sum \prod_{j=1}^r M_{i_j} \quad (9)$$

where

$$M_{i_j} = P_{i_j} / q_{i_j}; \quad W = \prod_{j=1}^m q_j \quad (10)$$

and the summation extends over all values of i_j such that

$$1 \leq i_1 < i_2 < \dots < i_r \leq m.$$

If for a fleet of m members we call $P_r(t) = {}_m S_r$, then the binomial expansion for the probability of cracks in the fleet with $(m-1)$ members will be

$$\left({}_{m-1} S_0 t^0 + {}_{m-1} S_1 t^1 + \dots + {}_{m-1} S_{r-1} t^{r-1} + {}_{m-1} S_r t^r + \dots + {}_{m-1} S_{m-1} t^{m-1} \right)$$

The similar expression for a fleet with m members will be $(P_m t + q_m)$ times the above expression and so the coefficient of t^r (which is ${}_m S_r$) will be given by

$${}_m S_r = P_m \left\{ {}_{m-1} S_{r-1} \right\} + q_m \left\{ {}_{m-1} S_r \right\} \quad (11)$$

i.e.

$${}_m S_r = q_m \left[M_m \left\{ {}_{m-1} S_{r-1} \right\} + \left\{ {}_{m-1} S_r \right\} \right] \text{ where } M_m = \frac{P_m}{q_m}.$$

Use of this simple recurrence relationship greatly eases the work involved in the calculation of ${}_m S_r (= P_r(t))$ by decreasing the number of operations required in the evaluation.

Now $P_i = g\{D_i\}$ from (6) and in conjunction with (5) gives

$$P_i = g\{f_i(t)\} \quad (12)$$

and so P_i is a function solely of time. Thus all combinations of P_1 must be functions of time and so it is possible to put

$$W(= q_1 q_2 \dots q_m) \equiv F(t) \quad (13)$$

Similarly the identity below can be written,

$$\sum \prod_{j=1}^r M_{i_j} \equiv K(t) \quad (14)$$

Then

$$P_r(t) = F(t) K(t) \quad (15)$$

7 SIMPLIFIED METHOD

Algebraically the derivation of $P_r(t)$ is simple. However the calculation of particular cases can involve a fair amount of work which can be reduced by simplification of the above treatment. The simplest approximation to make applies to equation (5).

$$D_i = f_i(t) \quad i = 1, 2 \dots m$$

If this is simplified by making

$$f_i(t) \equiv f_j(t) \quad \begin{array}{l} i = 1, 2 \dots m \\ j = 1, 2 \dots m \end{array} \quad (16)$$

the effect is to assume that all the aircraft in the fleet under consideration have consumed the same amount of damage at one particular time and that they use up their fatigue life at the same rate as one another.

Putting equation (16) in equation (13) gives:-

$$W = \prod_{i=1}^{i=m} [1 - g\{f_i(t)\}] = [1 - g\{f(t)\}]^m = F(t) \quad (17)$$

Whilst (16) in (14) results in:-

$$K(t) = \sum \prod_{j=1}^r M_{i_j} = {}^m C_r \left[\frac{g\{f(t)\}}{1 - g\{f(t)\}} \right]^r \quad (18)$$

Let $g\{f(t)\} = u(t)$. Then substituting (17) and (18) in (15) gives:-

$$P_r(t) = F(t) K(t) = {}^m C_r u(t)^r \{1 - u(t)\}^{(m-r)} \quad (19)$$

This value $P_r(t)$ is the probability of exactly r cracks, no more no less. The probability of the number of cracks being any number up to and including r is given by $P_R(t)$ where

$$P_R(t) = \sum_{r=0}^{r=R} P_r(t) \quad (20)$$

One further approximation also simplifies the calculations considerably. The rate of damage accumulation by the aircraft in a fleet in one particular role is approximately constant and so equation (16) is further reduced to

$$D = c t$$

where c is the rate of accumulation of fatigue damage index and t is the time measured from the introduction of the aircraft into service. The effect on Fig.6 is that the horizontal scale can now be replaced by a linearly proportional time-scale.

Sometimes a more useful probability than $P_R(t)$, the probability of any number of cracks up to and including r , is $Q_R(t)$, the probability of there being more than r cracks. $Q_R(t)$ is then given by the expression

$$Q_R(t) = 1 - \sum_{r=0}^{r=R} P_r(t) = 1 - P_R(t) \quad (21)$$

Fig.7 shows the plot of $P_R(t)$ and $Q_R(t)$ against $\frac{\log y}{\log S}$ obtained by using the example of Fig.3 with the further assumption of a fleet size of 20 aircraft and the approximate method of calculation. It was assumed for simplicity that there was only one part liable to fatigue failure on each aircraft. To maintain as much generality as possible in this graph the horizontal scale has been

plotted as $\frac{\log y}{\log S}$. The conversion from this scale to a fatigue damage index scale and thence to a time-scale would require the assumption of values for \bar{X} , C and the date of entry into service and so has been omitted.

8 COMPARISON OF THE TWO METHODS OF ANALYSIS

As a basis for the comparison of the accurate and the approximate methods of analysis an imaginary fleet was constructed consisting of 20 aircraft to each of which was assigned a fatigue damage index at the start of the period considered (Table 1). A rate of consumption of the fatigue damage index of 10 per annum and a mean fatigue damage index of 100 to the start of the crack were assumed. This latter value was chosen at 100 to enable to be made an easy appreciation of the position of each aircraft with respect to the mean of the distribution of times at which the crack under consideration could start.

Similarly it was assumed that only one full-scale fatigue test had been conducted, consisting of the cracking of the same component in both of two wings, that the wing was the part, liable to failure by fatigue, under consideration and naturally that a crack in either wing had to be repaired. This latter point must be emphasized for equation (1) will give the probability of cracking of a single wing (a say). The probability of non-cracking of a single wing is thus $b = 1-a$.

The probability of cracking in an aircraft with two wings present and essential, at a corresponding life, is thus given by $a^2 + 2ab$, whilst the probability of non-cracking is $1 - a^2 - 2ab$ or b^2 . These then are the values for p_i and q_i to be used in our equation (9), i.e.

$$\begin{aligned} p_i &= a^2 + 2ab \\ q_i &= b^2 \end{aligned}$$

A computer programme for the Manchester University Atlas Computer was written to produce values of $P_r(t)$ according to equation (10) when supplied with known values of the mean of the crack distribution, the individual aircraft fatigue damage indices, the number of aircraft in the fleet, the number of full-scale tests conducted, the rate of consumption of the fatigue damage index, the period of life to be covered and the value of the coefficient of

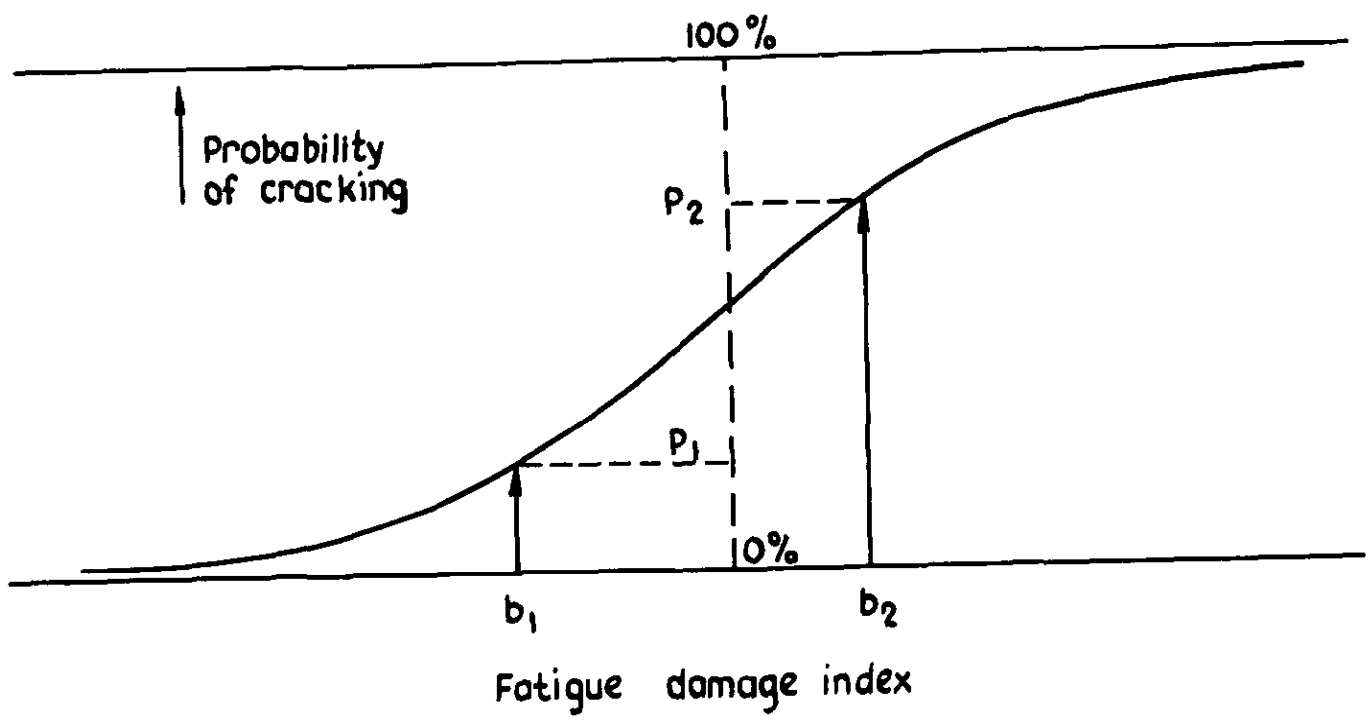


Fig.6

It is also noticeable that as the fatigue damage index increases from year to year, so the agreement between the two methods becomes better. The normalized deviates from the mean of the failure distribution become smaller as the fatigue damage index increases and so the individual probabilities are less sensitive to the small difference in life consumed. Thus the assumption of a universal fatigue damage index causes less error when the damage indices of the individual aircraft approach the mean of the failure distribution.

In service the coefficient of variation would be more likely to be in the range 0.05-0.10 rather than in the range 0.10-0.50 (in terms of logarithms to the base of 10) and from the Figs.8-10 it is noticeable that the discrepancies were increasing as the coefficient of variation decreased. The spread in the individual fatigue damage indices is rather high compared to the magnitude of the individual fatigue damage indices at the beginning of the period considered, but towards the end of this period when the magnitude of the indices has increased the spread is more realistic in relation to the indices, and the discrepancies so caused should be of the correct order.

9 CONCLUSIONS

Inaccuracies introduced by the adoption of an approximate method of analysis, whereby an average accumulation of fatigue damage index is assumed throughout the fleet, increase with a decrease in the coefficient of variation of the failure distribution and with a decrease in the fatigue damage index accumulated. However a decrease in the spread of the individual fatigue damage indices results in an increase in the accuracy of the approximate method. The discrepancies between the results obtained from the approximate and accurate methods are small compared with those caused by the introduction of the arbitrary conversion factor of 2 from test mean life to service mean life, for which there is no substantial evidence. It is felt that the approximate method as outlined above is an acceptable alternative to the accurate method, and that perhaps a better approximation would be given by taking either the arithmetic or geometric mean of the individual aircraft crack occurrence probabilities rather than the mean of the individual aircraft fatigue damage indices.

Acknowledgement

The author is pleased to have this opportunity to acknowledge with thanks the suggestion by Mr. M. G. Cox of the Division of Numerical and Applied Mathematics, N.P.L., that a recurrence relationship might be used in the evaluation of the probability on page 8 of this Paper.

Table 1

CHARACTERISTICS OF FICTITIOUS FLEET AND ITS
ASSOCIATED FAILURE DISTRIBUTION

Individual aircraft fatigue damage indices			
10	20	26	31
13	22	26	33
16	23	27	34
17	24	28	37
19	24	30	40

Mean = 25

Rate of aircraft fatigue damage index consumption = 10 per annum

Mean of distribution of crack-start indices = 100

Coefficient of variation of crack-start distribution

(on a logarithm to the base 10 and in terms of the fatigue damage index)

0.05; 0.10; 0.50; 1.00

4 separate cases

SYMBOLS

A	fatigue test factor (by which the full-scale test mean result must be divided to obtain the time for starting inspections)
D_i	fatigue damage accumulated by the general aircraft (i)
$f_i(t)$	function of time defining the fatigue damage accumulated by aircraft (i)
$g(D)$	function of the damage accumulated by an aircraft defining the probability of occurrence of a crack in that aircraft (P_i)
L_I	fatigue damage index at which the probability of occurrence of the crack considered is 1 in 740, i.e. the index at which inspections for this crack should start
m	number of aircraft in fleet
M_i	$\frac{P_i}{q_i}$
n	number of fatigue critical items full-scale fatigue tested
N	fatigue life of a specimen
P_i	probability of occurrence of crack in general aircraft i
$P_r(t)$	probability of exactly r occurrences of crack up to time t
$P_R(t)$	probability of any number of cracks up to and including r
q_i	probability of non-occurrence of a crack in general aircraft i
$Q_R(t)$	probability of there being more than r cracks
\bar{S}	service mean fatigue index rating for the occurrence of a crack
t_i	individual fatigue index ratings of times to occurrence of crack on separate tests
\bar{T}	test mean fatigue index rating for the occurrence of a crack
x	strength of randomly selected specimen, logarithm of life, = $(\log N)$
\bar{x}	estimated mean strength of all specimens = $(\overline{\log N})$, = $(\log \bar{T})$
V	coefficient of variation of life to cracking distribution on \log_{10} basis
W	$q_1 q_2 \dots q_m$
Z	standardized normal deviate

REFERENCES

<u>No.</u>	<u>Author</u>	<u>Title, etc.</u>
1	E. Marjorie Owen	The probability of fatigue failure of aircraft if the fatigue life is exceeded. R.A.E. Technical Note Structures 287 A.R.C. 22 439, August 1960
2	N. I. Bullen	A note on test factors. A.R.C. R & M 3166, September 1956

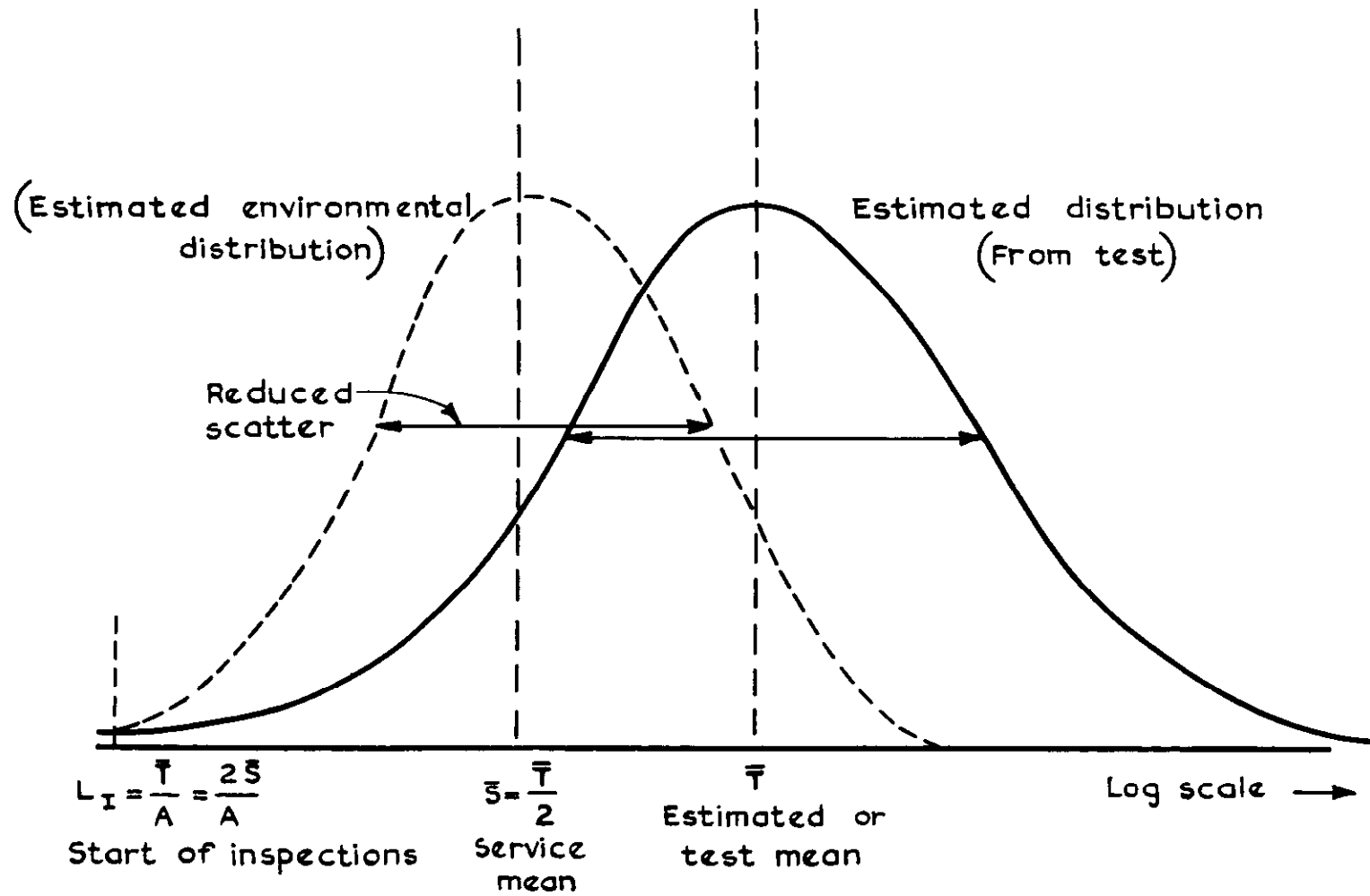


Fig. 1

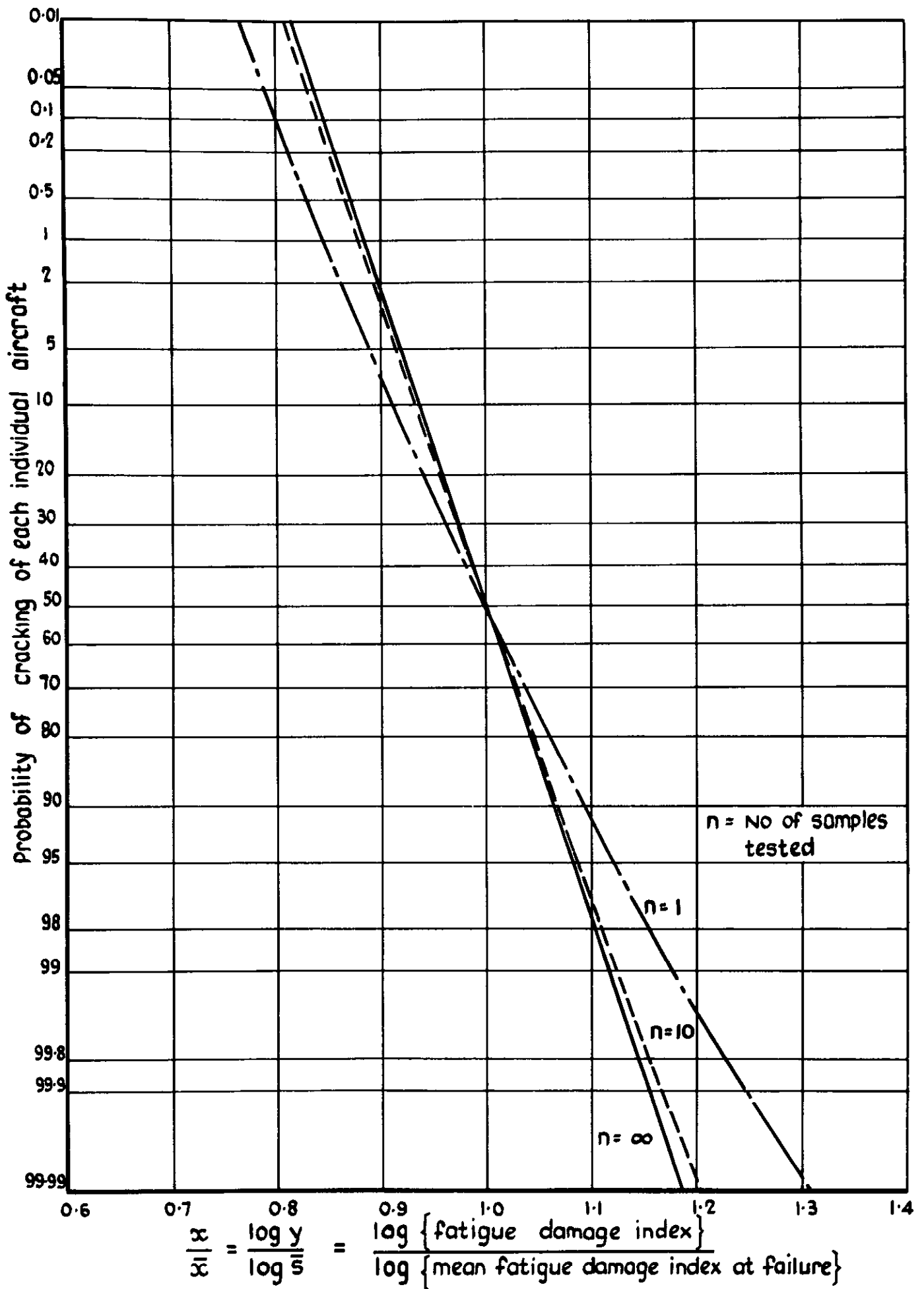


Fig. 2 Plot of Geary statistic; v assumed known = 0.05

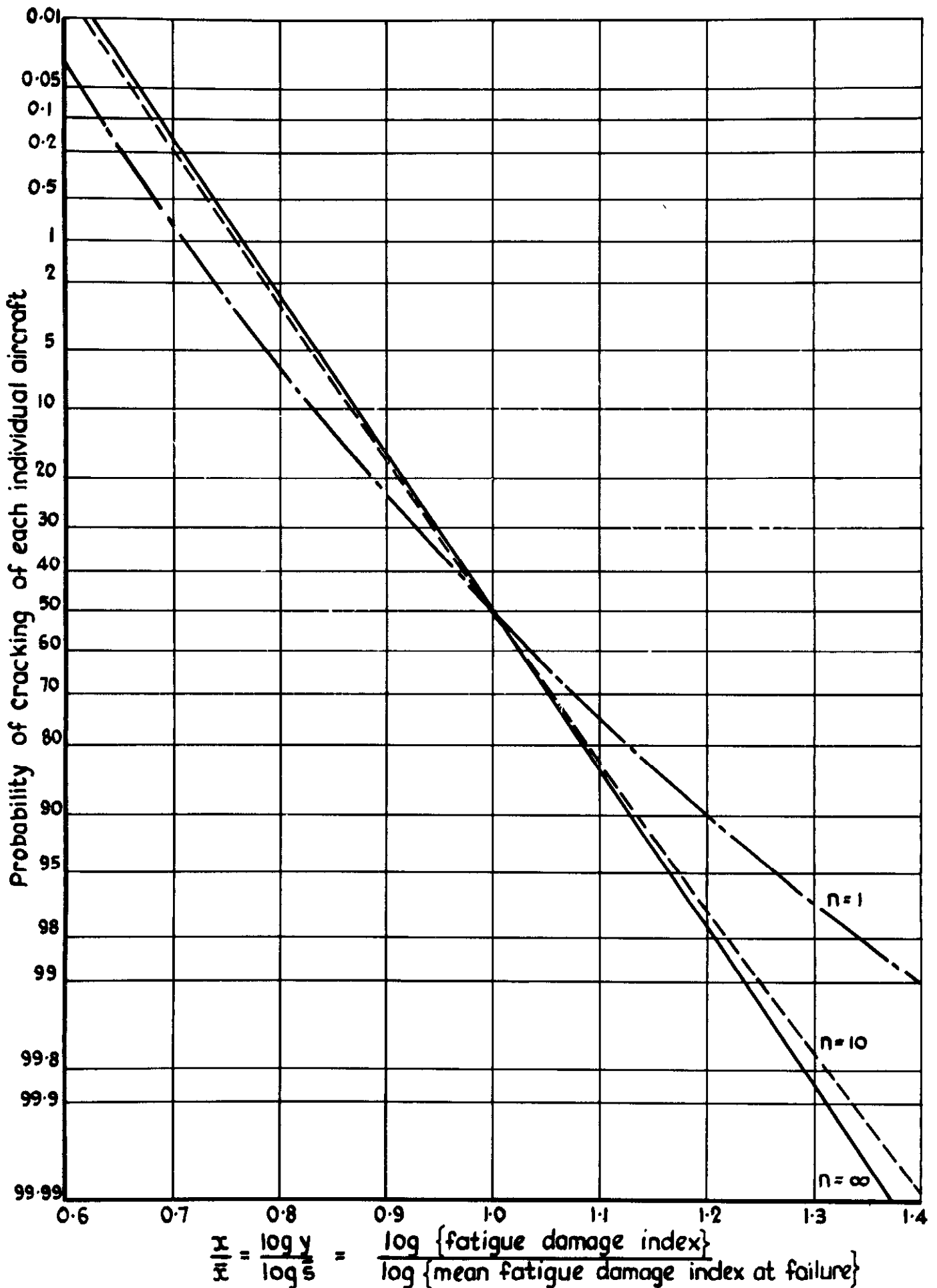


Fig.3 Plot of Geary statistic; v assumed known = 0.10

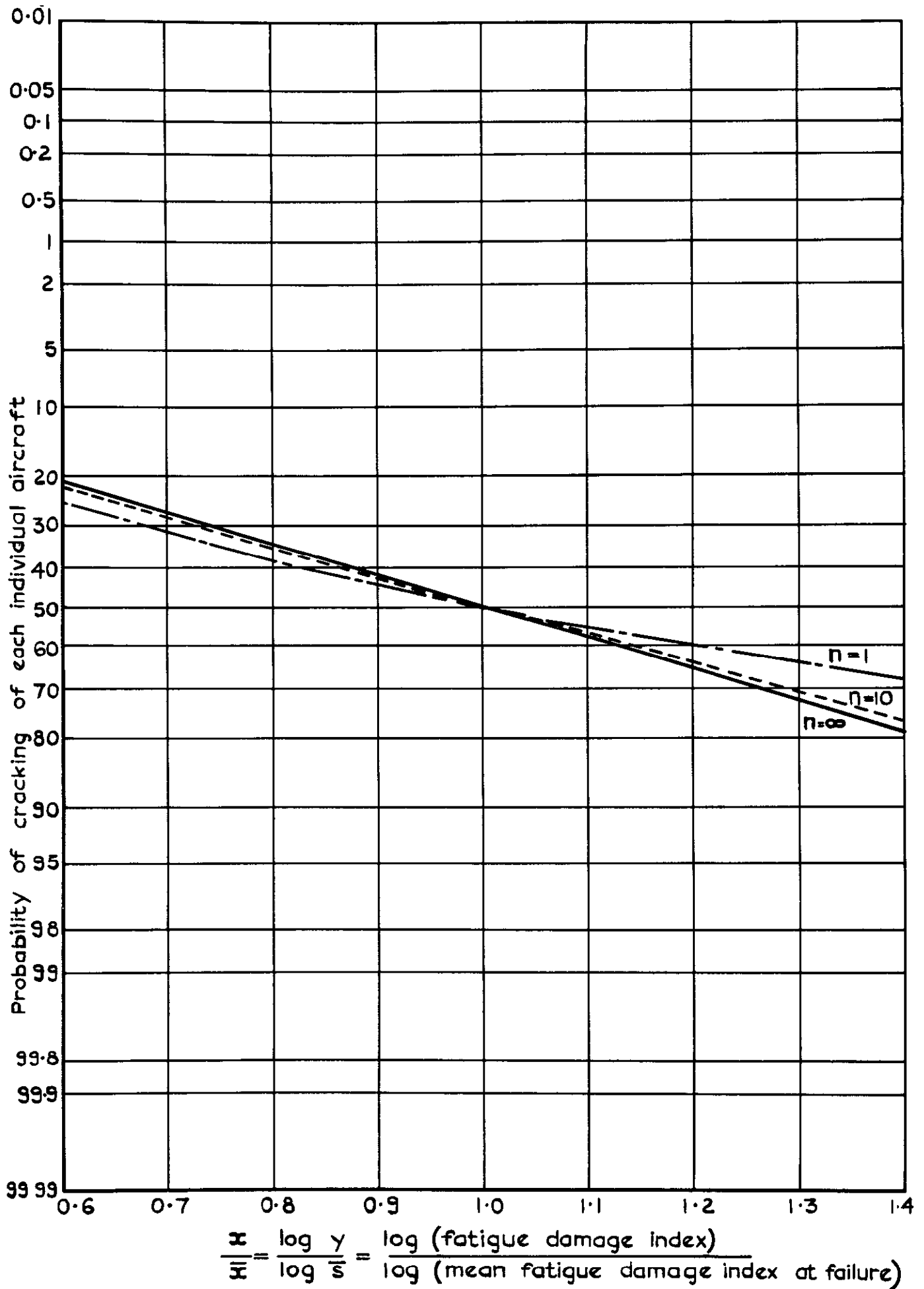


Fig. 4 Plot of Geary statistic; v assumed known = 0.50

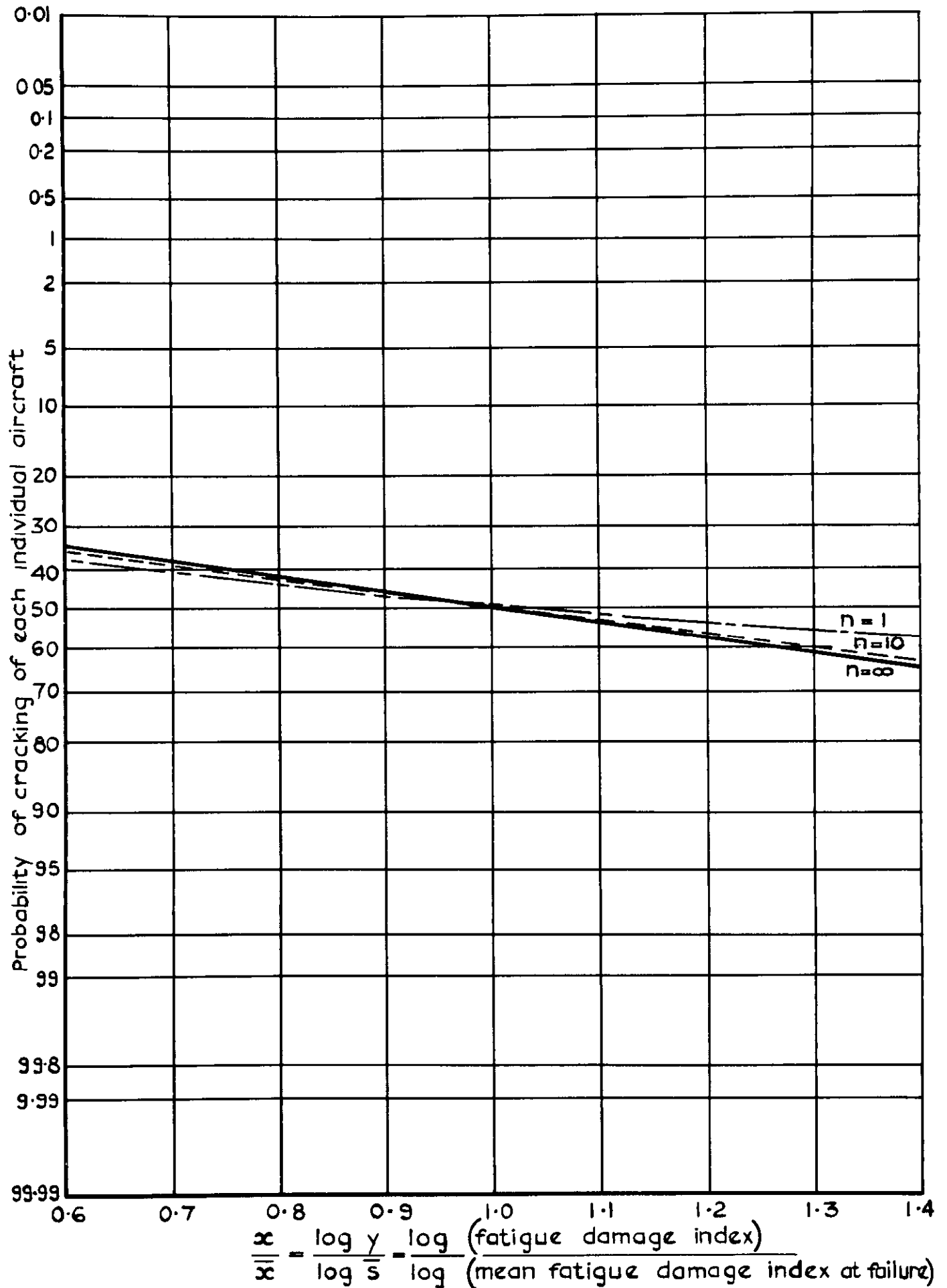


Fig. 5 Plot of Geary statistic; v assumed known = 1.00

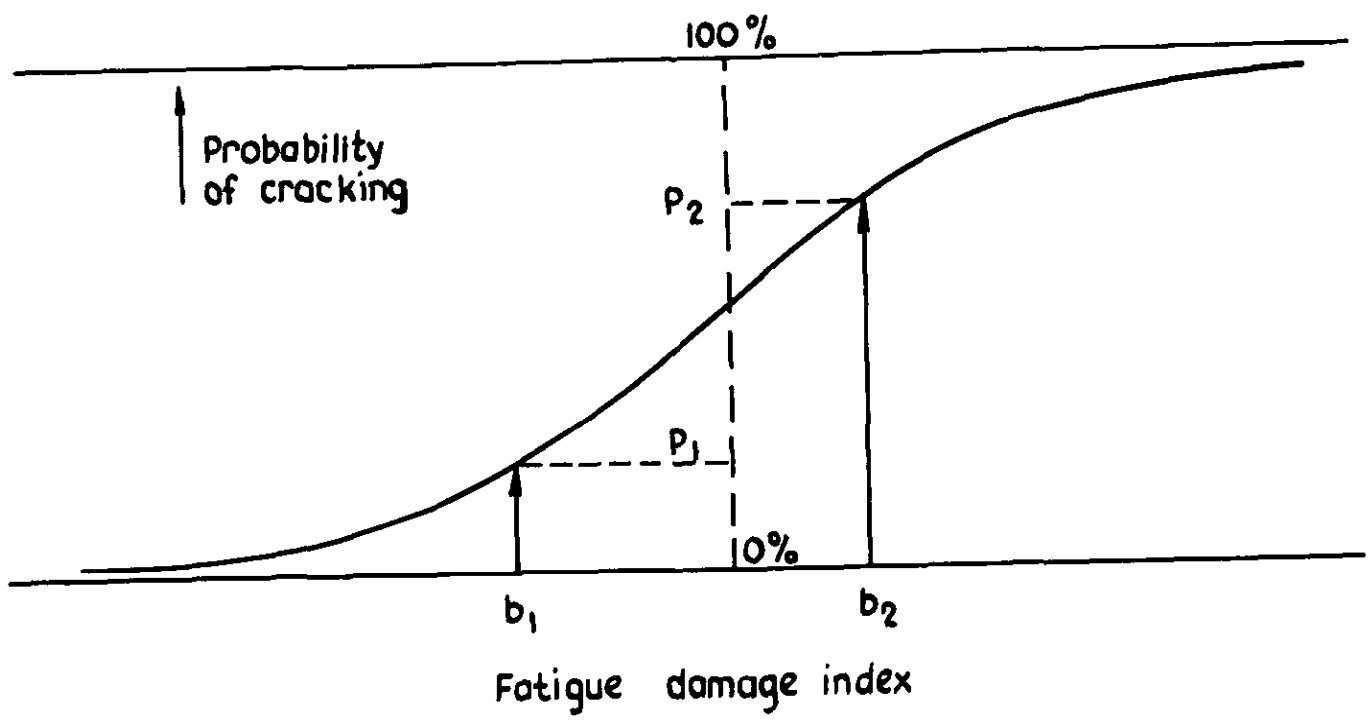
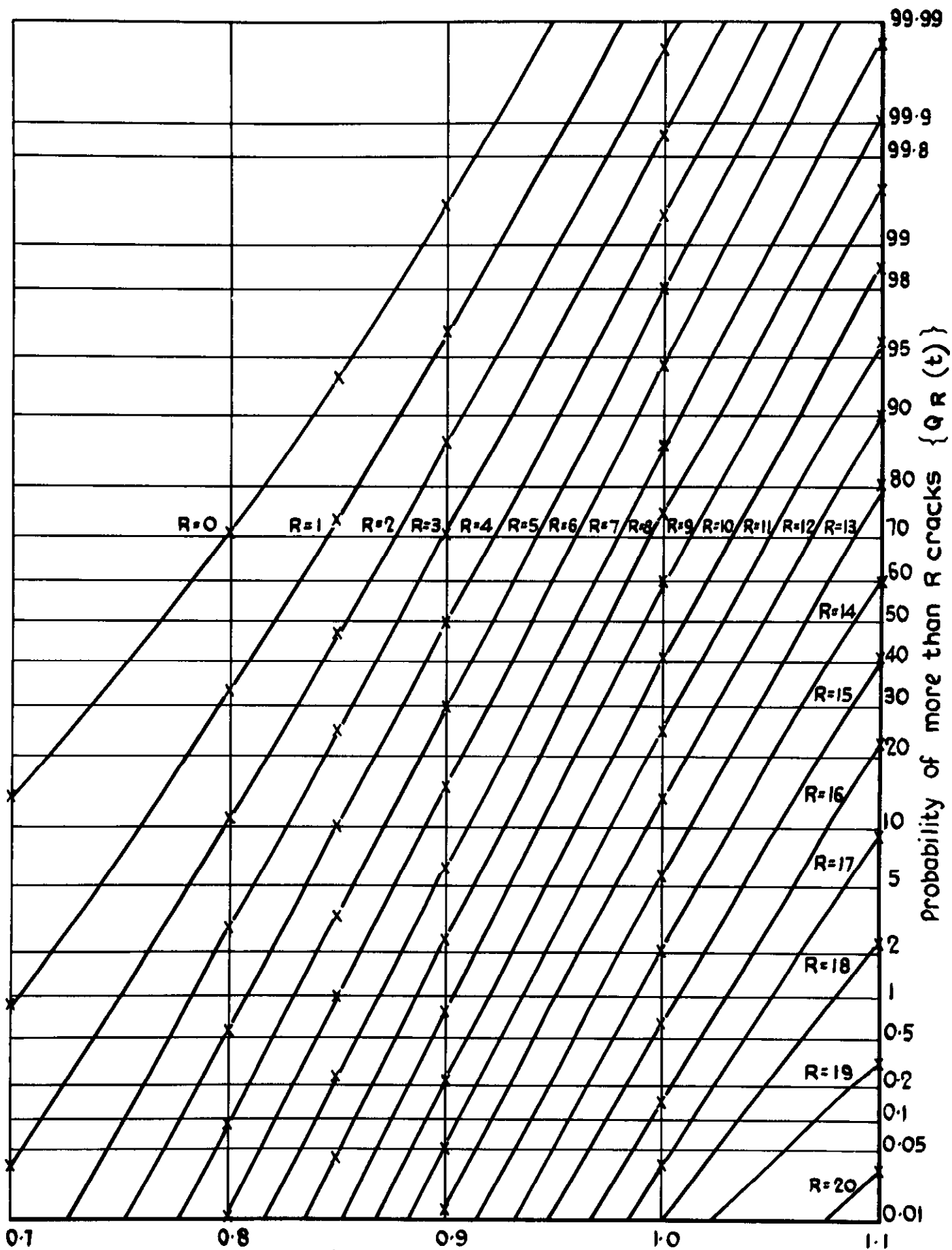


Fig.6



$$\frac{\log y}{\log \bar{s}} = \frac{\log\{\text{fatigue damage index}\}}{\log\{\text{mean fatigue damage index at failure}\}}$$

Fig.7 Fleet of 20 aircraft ; coefficient of variation = 0.10

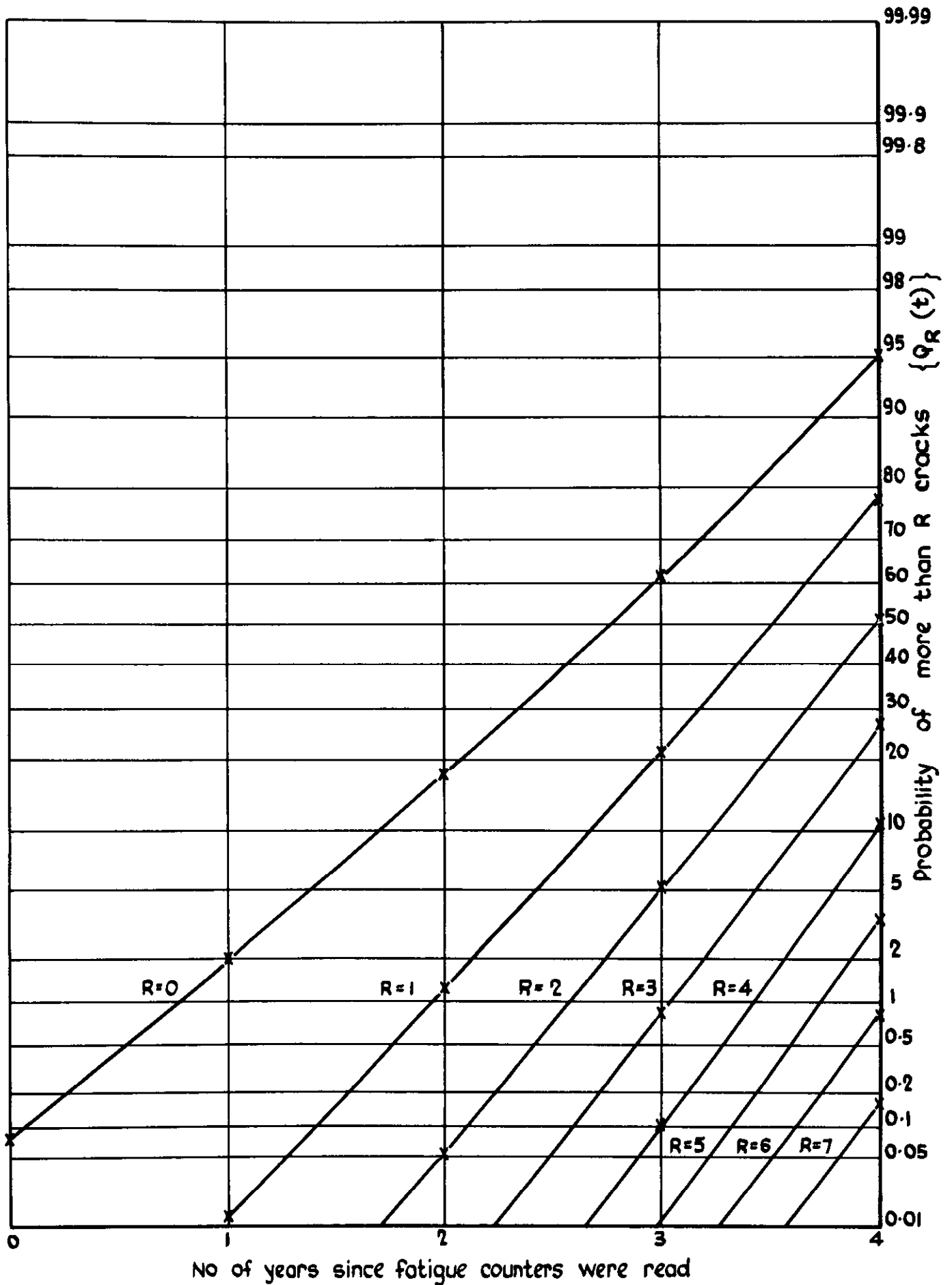


Fig. 8a $V = 0.05$ Accurate method

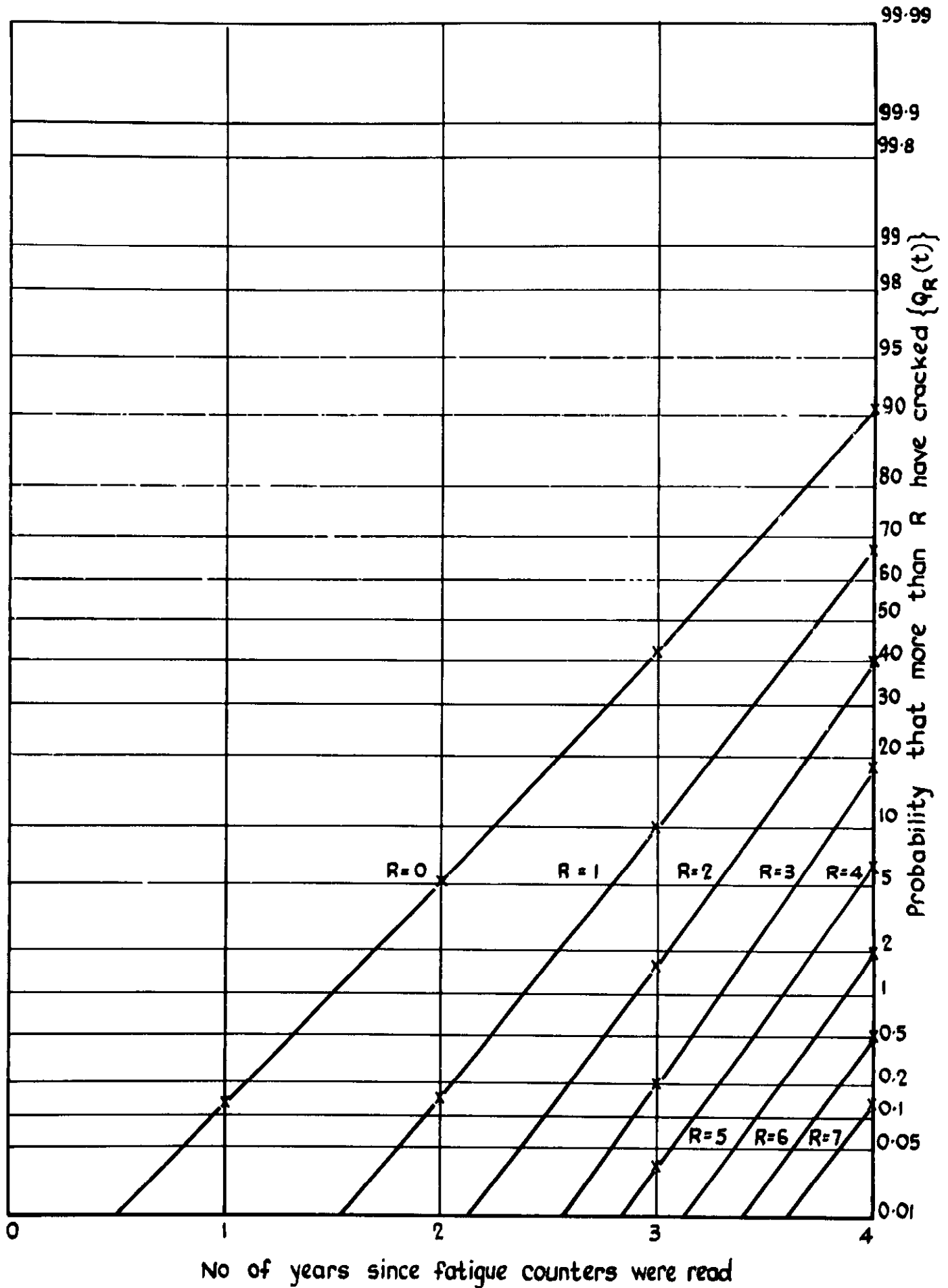


Fig. 8b $V = 0.05$ Approximate method

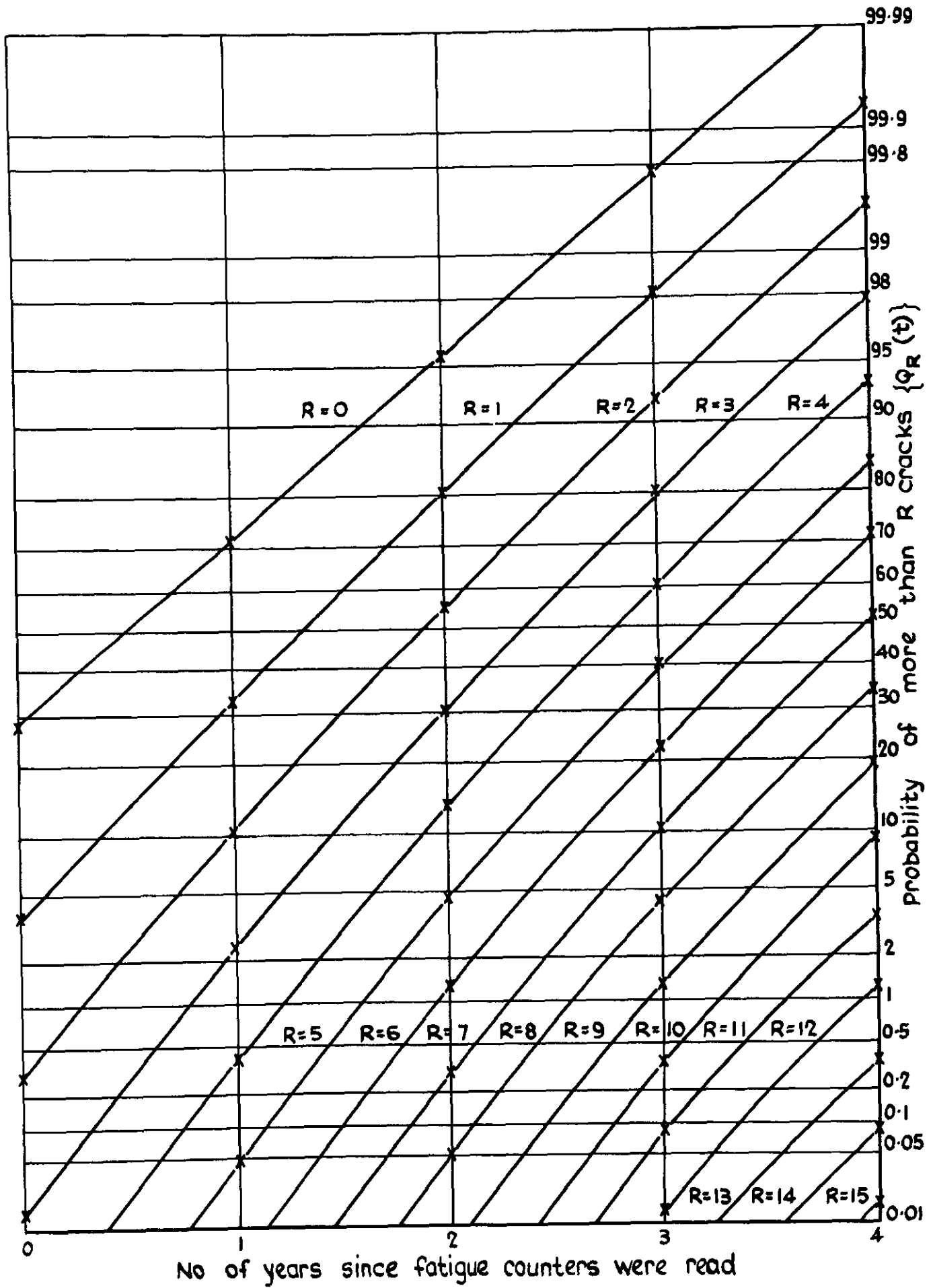


Fig. 9a $V = 0.10$ Accurate method

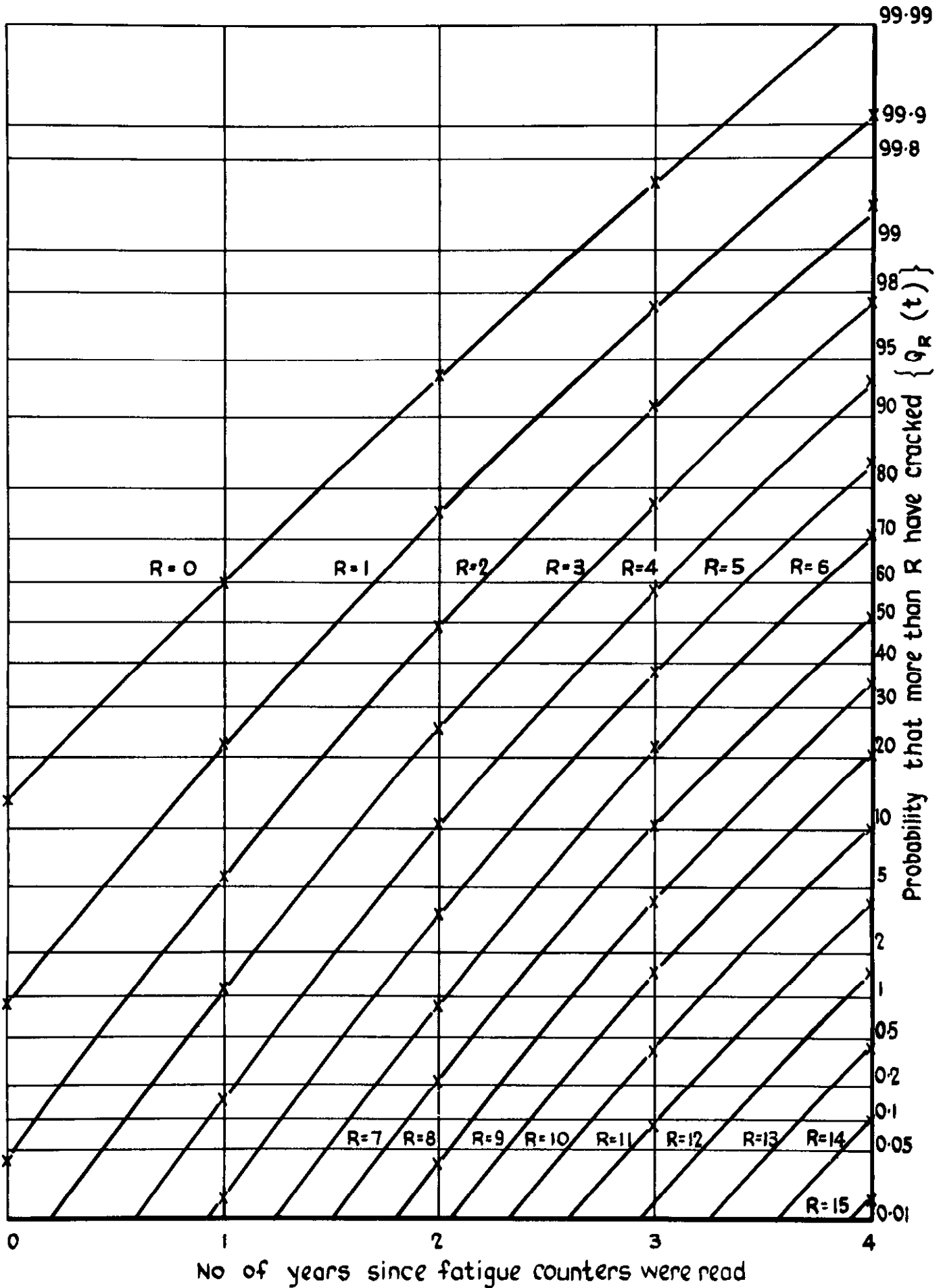


Fig. 9b $V = 0.10$ Approximate method

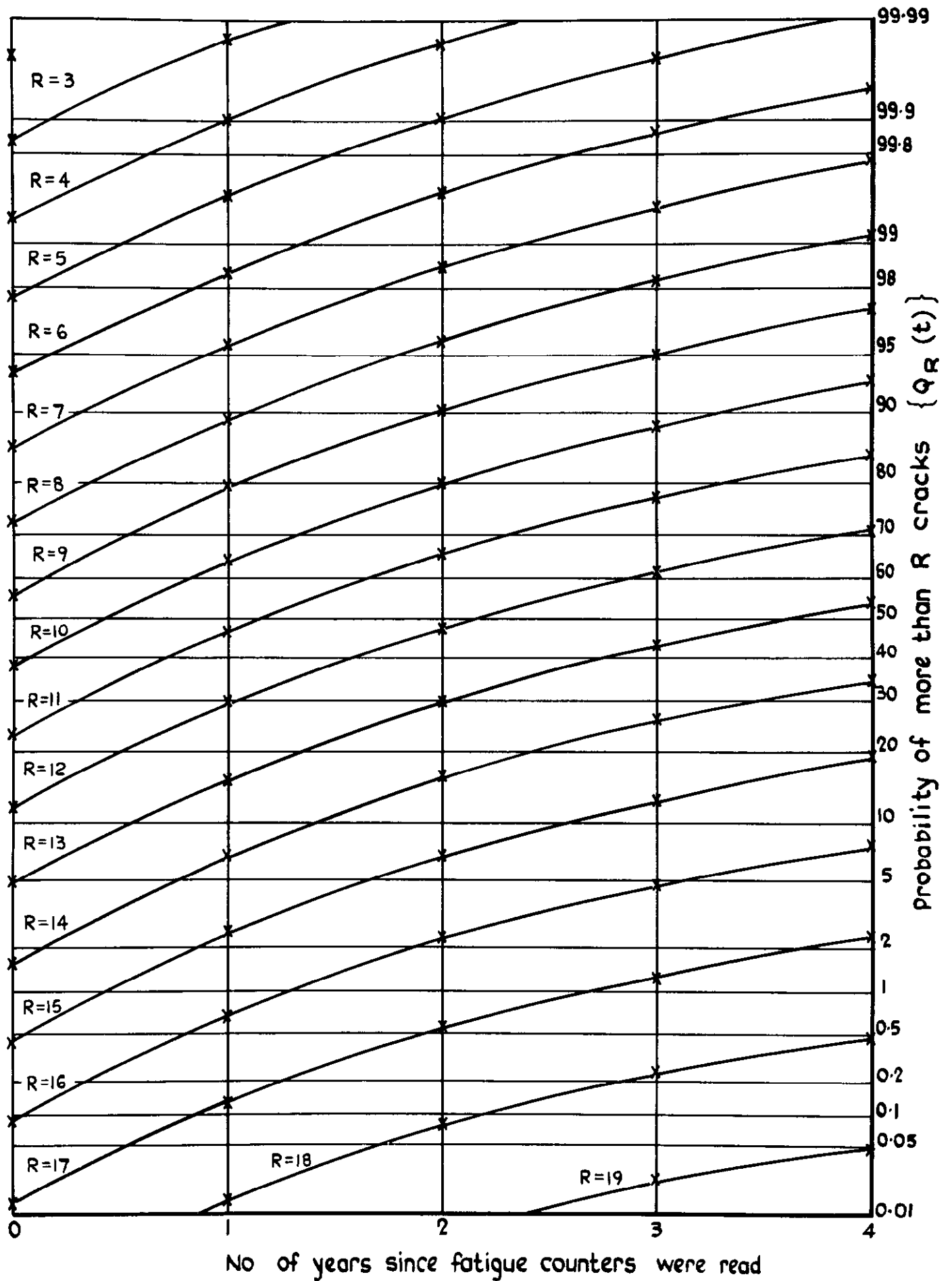


Fig 10a $V = 0.50$ Accurate method

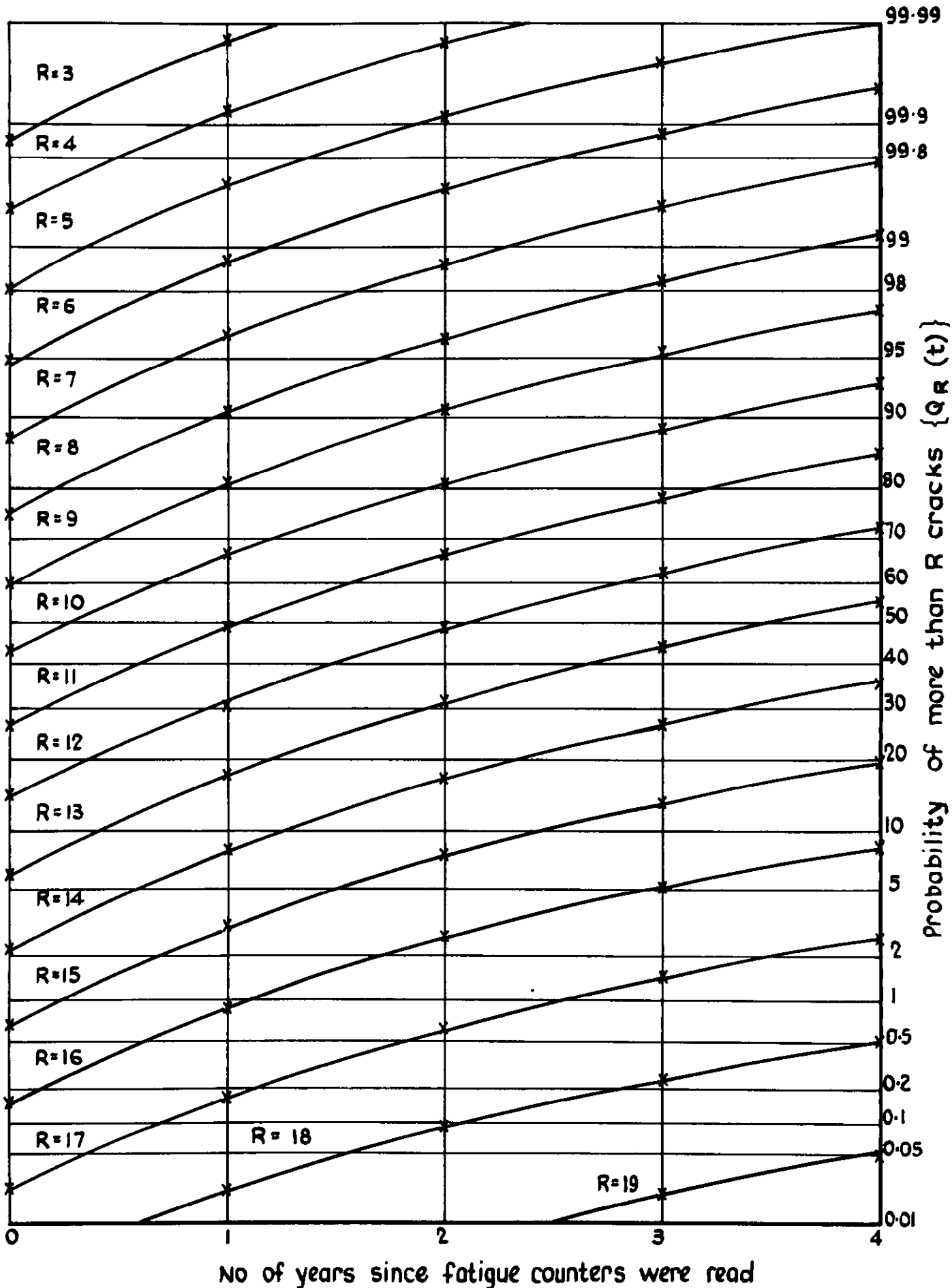


Fig. 10b $V = 0.50$ Approximate method

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