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Some Comments on the Conditions in a Local Supersonic Flow Region

By

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SUMMARY

The paper sets out to summarize the properties of the flow in a local supersonic, two-dimensional, steady potential flow region. Starting from the results of the theory of characteristics, the concept of wave strength is introduced and used to develop logically the properties of the supersonic region.

The conditions which must be imposed on the flow in order that it shall remain irrotational are reviewed. The practical significance of this is mentioned.

CONTENTS

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* Replaces NPL Aero Report 1169 (Revised) - A.R.C.27 322 (Revised)

Notation

P	pressure
ρ	density
a	sound speed
q	speed
θ	flow direction
M	Mach number ($= q/a$)
$\lambda \equiv$	q/a^* where a^* is the sound speed when local $M = 1$
s,n	streamline coordinates
x,y	general rectangular coördinates
ξ,η	characteristic coordinates
μ	Mach angle $\left(= \sin^{-1} \frac{1}{M}\right)$
ω	Prandtl-Meyer angle (see equation (6a))
γ	ratio of specific heats
$\tau =$	$\theta - \omega$ $\qquad \sigma = \theta + \omega$
$T =$	$-(\theta_s - \omega_s)$ $\qquad S = -(\theta_s + \omega_s)$
$\alpha =$	$\theta + \mu$ $\qquad \beta = \theta - \mu$
ε	angle between streamline and isobar
$\nu =$	$90 - \varepsilon$
J,j	transformation Jacobians
f	acceleration

Suffices denote partial differentiation

1./

1. Introduction

When a mathematical model is posed as a representation of a physical phenomenon it needs investigating to determine to what extent this representation is correct. Provided that the model has been correctly posed (i.e., has a unique solution, etc.) then its value is best estimated by comparing its predictions with reality.

In the problem of the transonic flow past an aerofoil the usual assumptions made are that the flow is two-dimensional, steady and irrotational. The purpose of the present work is to summarize the properties of the flow obtained from this model, and to investigate conditions for flow breakdown. There is considerable practical importance associated with the phenomenon of potential flow breakdown (shock wave formation in a real flow) since this occurrence results in a large increase of aerofoil drag (wave drag). The need exists to understand the mechanism of flow breakdown in order to establish optimum operating conditions.

The basic theoretical concept used in the analysis is the method of characteristics from which the properties of the flow model follow naturally. We should not, however, expect the properties of the flow model to necessarily agree with experimental findings. Where disagreement is found the flow model should be modified to improve the agreement. In general, recourse must be made to experiment in order that the deficiencies of the flow model may be rectified, the type of experiment needed being qualitative rather than quantitative.

For the present a complete comparison of the flow model with experiment is not undertaken but doubts about its adequacy are raised.

2. The Theory of Characteristics Applied to a Local Supersonic Flow Region

The flow under consideration is one where the local supersonic flow region is bounded on one side (along a streamline) by a solid surface while the rest of the boundary is the subsonic main stream - i.e., the $M = 1$ isobar. The flow is assumed throughout to be steady, isentropic, irrotational and two-dimensional.

The general theory of the method of characteristics is not repeated here since it is adequately treated in such standard works as Refs. 1 and 2. The pertinent results for the type of flow considered here are stated below and then the concept of a wave strength is developed to give some insight into the structure of such a flow region.

2.1 Results from the theory of characteristics

2.1.1 Referring to fig. (1), we define the characteristics ξ, η to be inclined at the Mach angle μ to the streamline 's'; the streamline is at an angle 'θ' to the reference direction 'x'.

If we define $\lambda = q/a^*$, the equations of motion along the streamline are obtained as follows.

Expressions/

Expressions for the normal and tangential accelerations are given in Ref. 3 as:

$$\left.\begin{aligned} P_n &= -\rho q^2 \theta_s , \\ P_s &= -\rho q\, q_s , \end{aligned}\right\} \qquad \ldots (1)$$

where, as throughout, suffices denote partial differentiation.

For irrotational flow

$$q_n = q\, \theta_s , \qquad \ldots (1a)$$

while continuity demands that

$$\frac{\partial}{\partial s}(\rho q) = -\rho q\, \theta_n ,$$

or, on using equation (1) together with the relation $P_s = a^2 \rho_s$,

$$q_s = \frac{q\, \theta_n}{M^2 - 1} . \qquad \ldots (1b)$$

In terms of the variable λ, equations (1b) and (1a) become

$$\left.\begin{aligned} \frac{\lambda_s}{\lambda} &= \frac{\theta_n}{M^2 - 1} , \\ \frac{\lambda_n}{\lambda} &= \theta_s . \end{aligned}\right\} \qquad \ldots (2)$$

For isentropic flow we have, from Bernouilli's equation,

$$\lambda^2 = \frac{\frac{\gamma+1}{2} M^2}{1 + \frac{\gamma-1}{2} M^2} , \qquad \ldots (3)$$

giving

$$d\lambda = \frac{\lambda}{M} \frac{1}{1 + \frac{\gamma-1}{2} M^2} \, dM. \qquad \ldots (3a)$$

Considering the variation of quantities along the characteristics, the following fundamental facts may be noted; they apply to all flows of the type under consideration (steady, two-dimensional, potential flow).

Treating first the ξ family, we note that

$$\left.\begin{aligned} \lambda_\xi &= \lambda_s \cos\mu + \lambda_n \sin\mu , \\ \theta_\xi &= \theta_s \cos\mu + \theta_n \sin\mu . \end{aligned}\right\} \qquad \ldots (4)$$

Then/

Then since

$$\sin\mu = \frac{1}{M}, \quad \cos\mu = \frac{\sqrt{M^2-1}}{M}, \quad \tan\mu = \frac{1}{\sqrt{M^2-1}}, \qquad \ldots (5)$$

and writing

$$d\omega = \frac{\sqrt{M^2-1}}{\lambda}\, d\lambda , \qquad \ldots (6)$$

the equations (4) may be cast in the form

$$\left.\begin{aligned} \omega_\xi &= \frac{\sqrt{M^2-1}}{M}(\omega_s + \theta_s) , \\ \theta_\xi &= \frac{\sqrt{M^2-1}}{M}(\omega_s + \theta_s) , \end{aligned}\right\} \qquad \ldots (4a)$$

where use has also been made of the equations (2).

The result (4a) implies that

$$\theta_\xi - \omega_\xi = 0 ,$$

or

$$\theta - \omega = \text{const} \equiv \tau , \qquad \ldots (7)$$

along the ξ characteristic.

Analogous equations to (4a) for the η family are

$$\left.\begin{aligned} \omega_\eta &= \frac{\sqrt{M^2-1}}{M}(\omega_s - \theta_s) , \\ \theta_\eta &= -\frac{\sqrt{M^2-1}}{M}(\omega_s - \theta_s) , \end{aligned}\right\} \qquad \ldots (8)$$

giving

$$\theta_\eta + \omega_\eta = 0$$

or

$$\theta + \omega = \text{const} \equiv \sigma \qquad \ldots (9)$$

along the η characteristic.

In general σ and τ will be functions of s and n.

By using equation (3), equation (6) can be integrated to give

$$\omega = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1} \qquad \ldots (6a)$$

thus identifying ω with the Prandtl-Meyer function. With ω as the variable the equations (2) become

$$\left.\begin{aligned} \omega_s &= \frac{\theta_n}{\sqrt{M^2-1}} , \\ \omega_n &= \sqrt{M^2-1} \cdot \theta_s \end{aligned}\right\} \qquad \ldots (2a)$$

The following equalities may also be noted

$$\left.\begin{aligned} d\mu &= -\frac{1}{M\sqrt{M^2-1}}\,dM \\ &= -\left(\frac{1+\frac{\gamma-1}{2}M^2}{\sqrt{M^2-1}}\right)\frac{d\lambda}{\lambda} \\ &= -\left(\frac{1+\frac{\gamma-1}{2}M^2}{M^2-1}\right)d\omega\,. \end{aligned}\right\} \quad \ldots (10)$$

Also, since

$$d\omega = \frac{\sqrt{M^2-1}}{M\left(1+\frac{\gamma-1}{2}M^2\right)}\,dM\,,$$

it follows that at $M = 1$ $\frac{d\omega}{dM} = 0$, so that, for M_s finite, $\omega_s = 0$; hence in such figures as fig. (2) the $\theta \pm \omega$ curves should touch the θ curve at $M = 1$. For a non-zero θ_s this requires $T > 0$ in fig. (2). The significance of this will be seen in section 2.2.

If α, β are the inclinations of the ξ, η waves to the x direction, we have

$$\begin{aligned} \alpha &= \theta + \mu \quad \text{along a } \xi \text{ wave}\,, \\ \beta &= \theta - \mu \quad \text{along a } \eta \text{ wave}\,, \end{aligned} \quad \ldots (11)$$

or differentiating along the stream direction

$$\begin{aligned} \alpha_s &= \theta_s + \mu_s\,, \\ \beta_s &= \theta_s - \mu_s\,, \end{aligned} \quad \ldots (11a)$$

showing that for a compressive flow on a convex surface, for which

$$\theta_s < 0 \qquad \mu_s > 0\,,$$

then

$$\begin{aligned} \alpha_s &= |\mu_s| - |\theta_s|\,, \\ \beta_s &= -|\theta_s| - |\mu_s|\,, \end{aligned} \quad \ldots (12)$$

so that $\beta_s < 0$ and the η family converge.

On the other hand,

$$\alpha_s > 0 \quad \text{if} \quad |\mu_s| > |\theta_s|,$$

$$\alpha_s < 0 \quad \text{if} \quad |\mu_s| < |\theta_s|,$$

so/

so that the ξ waves can either converge or diverge, depending on the particular flow. This point is considered again later. - section (3.2.1).

Differentiation of equation (11) along the respective characteristics gives for the curvature of the characteristics:

$$\alpha_\xi = -\frac{2}{\sqrt{M^2-1}}\left(1-\frac{3-\gamma}{4}M^2\right)\frac{\lambda_\xi}{\lambda},$$

$$\beta_\eta = \frac{2}{\sqrt{M^2-1}}\left(1-\frac{3-\gamma}{4}M^2\right)\frac{\lambda_\eta}{\lambda}, \qquad \ldots (11b)$$

where the equations (4a), (6) and (10) have been incorporated. Thus, as shown by Laitone in Ref. 7, α_ξ and β_η change sign when $M^2 = \frac{4}{3-\gamma}$ $(M = 1\ 581)$ - it being shown in section 2·2 that $\lambda_\xi < 0$ and $\lambda_\eta > 0$. It is interesting to note that this inflexion point is not generally related to the inflexion point in the Prandtl-Meyer function. Since, by equations (3a) and (6), $\frac{d\omega}{dM} = \frac{\sqrt{M^2-1}}{M(1+\frac{\gamma-1}{2}M^2)}$, equating $\frac{d^2\omega}{dM^2}$ to zero gives:

$$M = +\left\{\frac{3}{4}+\frac{1}{4}\sqrt{\frac{9\gamma+7}{\gamma-1}}\right\}^{\frac{1}{2}}$$

as the only admissible solution of the resulting quartic equation. Hence only at a value of $\gamma = 1{\cdot}400$ do the two inflexion points occur at the same Mach number, i.e., $M = 1{\cdot}581$ (or $\lambda = \sqrt{2}$).

2.1.2 It is useful to define the strength of the characteristics (designated "wave strength"). The strength of a wave may be measured by its effect on waves of the other family. Thus λ_η (or ω_η) could be taken as the strength of a ξ wave*. However if use is made of equations (4a) and (8) a slightly different, but more useful, definition of wave strength emerges.

Differentiating equations (7) and (9) along the streamline gives

$$\left.\begin{aligned} \theta_s + \omega_s &= \sigma_s, \\ \theta_s - \omega_s &= \tau_s. \end{aligned}\right\} \qquad \ldots (13)$$

Combining/

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*As was done by Nieuwland in Ref. 19. Reyn in Ref. 4 defined the contribution of each characteristic to λ_s as the wave strength (i.e., quantities such as $\frac{1}{2\cos\mu}\lambda_\xi$ are considered).

Combining equation (13) with equations (4a) and (8) gives

$$\omega_\xi = \frac{\sqrt{M^2 - 1}}{M}\sigma_s, \qquad \omega_\eta = \frac{\sqrt{M^2 - 1}}{M}\tau_s. \qquad \ldots (14)$$

Now define

$$S = -\sigma_s = -(\theta_s + \omega_s) \text{ the strength of the } \eta \text{ wave,} \qquad T = -\tau_s = -(\theta_s - \omega_s) \text{ the strength of the } \xi \text{ wave.} \qquad \ldots (13a)$$

Then from the equations (4a), (6) and (8)

$$\lambda_\xi = -\frac{\lambda}{M}S, \quad \lambda_\eta = \frac{\lambda}{M}T, \quad \theta_\xi = -\frac{\sqrt{M^2 - 1}}{M}S, \quad \theta_\eta = -\frac{\sqrt{M^2 - 1}}{M}T. \qquad \ldots (14a)$$

With the above sign convention for the wave strength, $T > 0$ makes ξ an expansion wave and $S > 0$ makes η a compression wave.

The derivative of T along the ξ characteristic is

$$T_\xi = T_s\cos\mu + T_n\sin\mu$$

$$= (\omega_{ss} - \theta_{ss})\cos\mu + (\omega_{sn} - \theta_{sn})\sin\mu \text{ from the definition (13a) of } T.$$

Differentiating equations (2a) to obtain θ_{sn}, ω_{sn}, we find that

$$\left.\begin{aligned} T_\xi &= \frac{(\theta_s - \omega_s) \cdot M_s}{\sqrt{M^2 - 1}} = -T \cdot M_s\tan\mu. \\ \text{Similarly} \quad S_\eta &= -S \cdot M_s\tan\mu. \end{aligned}\right\} \qquad \ldots (13b)$$

Repeated differentiation of equation (13b) (for the ξ wave as example) gives:

$$\frac{\partial^n T}{\partial\xi^n} = T \cdot f\left(M, M_s, M_{ss} \ldots\ldots \frac{\partial^n M}{\partial s^n}\right).$$

Hence, if $T = 0$ at some point on the ξ wave it will be zero along the whole wave, since all the derivatives of T along ξ are then zero. (See Lemma of section 2.3.) In other words, an isolated zero of T is not

possible/

possible in a supersonic flow region (except, possibly, at the sonic line). Hence, it follows that S and T do not change sign along the respective characteristics, and, since they are positive at the sonic line (see section 2.2), they will be positive throughout the supersonic flow region.

It follows from equation (13b) that S and T increase or decrease along the characteristics depending upon the sign of $\partial M/\partial S$. For example, for expanding flow ($M_s > 0$), T decreases along the ξ wave while S decreases along the η wave (taking account of the direction of the elements $\Delta\xi$ and $\Delta\eta$ - Fig. (1b).

Equation (13b) shows that, in general, $\partial T/\partial\xi$ (or $\partial S/\partial\eta$) becomes infinite at the sonic line where $\mu = 90°$. The value of T (or S) will, in general, remain finite since $T = -\theta_s$ when $M = 1$.

We note in passing that equation (11a) may be written:

$$\alpha_s = 2\left(\frac{3-\gamma}{4} M^2 - 1\right)\frac{\theta_s}{M^2 - 1} - \frac{1 + \frac{\gamma-1}{2} M^2}{M^2 - 1} T$$

$$\beta_s = 2\left(\frac{3-\gamma}{4} M^2 - 1\right)\frac{\theta_s}{M^2 - 1} - \frac{1 + \frac{\gamma-1}{2} M^2}{M^2 - 1} S. \qquad \dots (11c)$$

2.2 The structure of a local supersonic flow region

In the following the properties of a local supersonic flow region - as shown in Fig. (1a) - are developed in a logical way. It should be noted that, in general, the results only hold in regions where the characteristics end on the sonic line.

Let ε be the inclination of a constant velocity line 'ℓ' to the streamline - see Fig. (1b) - then

$$\lambda_\ell = \lambda_s \cos\varepsilon + \lambda_n \sin\varepsilon = 0,$$

so that

$$\tan\varepsilon = -\frac{\lambda_s}{\lambda_n}. \qquad \dots (15)$$

Combining equations (2), (2a) and (6) with equation (13a) yields

$$\lambda_s = \frac{\lambda}{2\sqrt{M^2 - 1}}(T - S),$$

$$\lambda_n = -\frac{\lambda}{2}(T + S), \qquad \dots (16)$$

and/

and

$$\theta_s = -\left(\frac{T + S}{2}\right),$$

$$\omega_s = \frac{T - S}{2}, \qquad \ldots (16a)$$

$$\theta_n = \frac{\sqrt{M^2 - 1}}{2}(T - S).$$

Equation (15) becomes

$$\tan \varepsilon = \tan \mu \left(\frac{T - S}{T + S}\right). \qquad \ldots (15a)$$

From equation (15a), Busemann's result - Ref. 5 - can be recovered, namely:

If T and S (in the present notation) are of the same sign, then $\varepsilon < \mu$, whereas if T and S are of opposite sign then $\varepsilon > \mu$. Busemann claims that the fact that this result must be true on the sonic line shows that T and S must be of the same sign. This does not follow directly, as above, from equation (15a) since at the sonic line $\mu = 90°$.

We argue, instead, that in general the sonic line is not perpendicular to the streamline, so that from equation (15a) we must have

$$T - S = 0 \text{ at the sonic line} \qquad \ldots (17)$$

and the sonic line slope is indeterminate from equation (15a). Busemann's result is thus recovered provided that the sonic line is not perpendicular to the streamline. If $\varepsilon = 90°$ then equation (15a) yields no information concerning S and T. Equation (17) also follows from the definitions given in equation (13a) and the fact that $\omega_s \equiv 0$ at the sonic line.

With the present choice of axes we must have at the sonic line

$$\lambda_n < 0,$$

and since T and S are to be of the same sign, it follows from equation (16) that

$$T > 0 \text{ and } S > 0. \qquad \ldots (17a)$$

Also, from equation (16) (using equation (17a)) it follows that when

$$\lambda_s > 0, \qquad T > S,$$

and when

$$\lambda_s < 0, \qquad T < S.$$

Since/

Since both S and T are positive, the ξ waves are expansion waves and the η waves must be compression waves. With $T > 0$ we have from equation (13a)

$$\tau_s < 0 \qquad \dots (17b)$$

along a streamline. Since $\tau_s = \theta_s - \omega_s$, and $\omega_s = 0$ when $M = 1$ (see remark after equation (10)), it follows that $\theta_s < 0$ when $M = 1$. Now θ_n is then zero (equation (2)), so that the rate of change of θ along the sonic line is simply $\theta_s \cos \varepsilon$. But it can easily be shown from equation (15) that ε must be an acute angle, and it follows at once that the flow direction decreases monotonically on moving along the sonic line, as was shown by Nikolski and Taganov[8].

For T and S to be positive, a compression wave must be a compression wave along its whole length (and similarly for expansion waves) and so a characteristic can only have one end on the sonic line, as was pointed out by Guderley*[6].

It is of interest to note that from equation (17) it follows that the two waves that meet on the sonic line must be of the same strength. The relative strengths of the two waves meeting on the surface depends on the pressure gradient as seen from equation (16); if $\lambda_s > 0$ then $T > S$, i.e., the outgoing wave is stronger than the incoming wave.

With both T and S positive, equation (14) shows that $\omega_\xi < 0$ and $\omega_\eta > 0$; and equation (14a) gives

$$\begin{array}{ll} \lambda_\xi < 0 & \lambda_\eta > 0 \\ \theta_\xi < 0 & \theta_\eta < 0 \end{array} \qquad \dots (18)$$

as indicated by Laitone in Ref. 7 and Nikolski and Taganov in Ref. 8.

2.3 Simple wave flows

The general theory of characteristics for two independent variables solves two quasi-linear partial differential equations of the form

$$Au_x + Bu_y + Cv_x + Dv_y + E = 0$$

where u,v are the dependent and x,y the independent variables. The coefficients A ... E are functions of (u,v,x,y). If these coefficients are functions of (x,y) only, the equations are linear. Again if the equations are homogeneous (E = 0) and if A ... D are functions of (u,v) only (i.e., reducible equations) the hodograph transformation may be applied to interchange dependent and independent variables. The equations so formed will be linear.

The/

*The apparent anomaly caused by the presence of a sonic line in the boundary layer of a real flow disappears with the introduction of a vorticity term in the equations.

The hodograph transformation can only be applied if the Jacobian $J = \dfrac{\partial(u,v)}{\partial(x,y)} \neq 0$.

The special case of fluid flow, for which the equations are reducible and $J = 0$ identically over a region, gives rise to simple wave flow. The fact that $J = 0$ over a region implies that u and v are not independent and the whole of the region in the x,y plane corresponds to a curve in the u,v plane. Fuller details are given in Ref. 1.

When the equations of motion in streamline coordinates are used (equations (2)) the relevant Jacobian is

$$J = \frac{\partial(\lambda,\theta)}{\partial(s,n)}$$

giving

$$\lambda_s\theta_n - \lambda_n\theta_s = 0 \text{ for simple wave flow.}$$

From equation (2) it follows that

$$J = (M^2 - 1)\frac{\lambda_s^2}{\lambda} - \lambda\theta_s^2 = 0, \qquad \text{... (19)}$$

or

$$\theta_s^2 = \frac{(M^2 - 1)}{\lambda^2}\lambda_s^2$$

$$\equiv \omega_s^2 \text{ by equation (6).}$$

On integrating along a streamline it follows that

$$\theta \pm \omega = \text{constant} \qquad \text{... (19a)}$$

for plane simple wave flow.

Further properties of simple wave flow are worth noting. Putting the results of equations (16) and (16a) into (19) gives

$$J = -\lambda.T.S. \qquad \text{... (19b)}$$

Hence for $J = 0$ we must have either T or S vanishing.

As an example, consider the case when $T = 0$. Then from equation (14a)

$$\lambda_\eta = 0, \quad \theta_\eta = 0,$$

and so from equation (10) $\mu_\eta = 0$, showing that the η waves must be straight lines with velocity and flow direction constant along them. We note in passing that the pressure gradient is locally perpendicular to the characteristic and that the velocity component in this direction is sonic.

An/

An important result follows from the fact that the wave strengths are constant along characteristics in simple wave flow (section 2.1.2).

Lemma

If at any point in the flow field a wave of one family crosses a simple wave of the other family, then all the waves of the second family which cross the said wave of the first family must be locally simple waves.

For: if (say) $T = 0$ at some point it will be zero along the whole of the ξ wave in question. Similarly for the η family when $S = 0$.

Distinction should be drawn between a single ξ wave for which $T = 0$. The former implies that the Jacobian $J \doteq 0$ along a line in the flow field and this constitutes a branch line in the flow (see Ref. 9 or Ref. 4). Simple wave flow strictly only results in the latter case when $T = 0$ over a region where the above lemma still applies.

It should be noted that simple wave flow cannot exist up to a sonic line which is of finite length and not in the characteristic direction, since such waves are of constant Mach number, and some limiting characteristic of the other family must exist (see fig. (3) where BC is such a limiting characteristic). There is still doubt as to whether simple wave flow can exist in the region AB in fig. (3) (where AB is both the final characteristic and the sonic line - which is thus straight and perpendicular to the streamline). From equations (6), (13) and (14a) we find for the case $T = 0$ that

$$\lambda_\xi = \frac{2\sqrt{M^2 - 1}}{M} \lambda_s$$

so if λ_s is finite λ_ξ must approach zero for $M \to 1$. Nikolski and Taganov in Ref. 8 show that the characteristic AB of fig. (3) would have to be of infinite length, thus proving that simple wave flow in a finite region is impossible. A modified form of their proof is given in the Appendix.

To illustrate a simple wave compression the compressing flow around a circular profile (θ_s = const) is shown in fig. (4). In the example the flow from a Mach number of 1·2 was taken for a value of $\theta_s = 0{\cdot}01$.

3. Comments on Criteria for Potential Flow Breakdown

In the following section some comments are made concerning the various criteria that have been proposed for the breakdown of potential flow. As far as possible the criteria have been cast in a form to be consistent with the notation of section (2) so that any relation between the criteria is more easily seen.

3.1 Conditions for infinite acceleration

The finding of an infinite acceleration in the flow field has arisen from two different lines of approach, namely treatments in the flow plane and solutions by the hodograph method.

3.1.1/

3.1.1 Bickley in Ref. 10 - following Scherberg in Ref. 11 - found a condition for the appearance of an infinite acceleration in the flow. The analysis of Bickley is summarized below.

Taking rectangular coordinates with the 'x' axis in the direction of the pressure gradient, so that $\frac{\partial p}{\partial y} = 0$ locally, the equations of motion may be manipulated to show that the acceleration $f \equiv u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$ is given by

$$f = \frac{a^2}{u^2}\frac{u^2 + v^2}{u^2 - a^2} v_y . \qquad \ldots (22)$$

This gives Bickley's result that the acceleration becomes infinite when the velocity component along the direction of the pressure gradient becomes sonic, provided that the quantity v_y is non-zero.

Substituting $a = u$ in Bernoulli's equation

$$\frac{a^2}{\gamma-1} + \frac{u^2 + v^2}{2} = \frac{\gamma+1}{2(\gamma-1)} a^{*2}$$

and noting that $\frac{\gamma+1}{\gamma-1} a^{*2} = q_m^2$ - the maximum possible velocity, gives

$$\frac{u^2}{a^{*2}} + \frac{v^2}{q_m^2} = 1. \qquad \ldots (23)$$

Equation (23) represents Scherberg's[11] critical ellipse in the hodograph plane and any flow whose streamline in the hodograph plane crosses this ellipse must attain infinite acceleration. The critical ellipse is shown on fig. (5) for a typical case. Since the critical ellipse is defined relative to the direction of the local pressure gradient, it is not a fixed curve in the hodograph plane.

On making the substitution

$$u = q \cos \nu, \qquad v = q \sin \nu,$$

where ν is the angle between the streamline and the direction of the pressure gradient, equation (23) can be cast in the form

$$M^2 \cos^2 \nu = 1, \qquad \ldots (23b)$$

where use has been made of equation (3).

We note that since $\frac{1}{M} \equiv \sin \mu$ equation (23b) shows that

$$\cos^2 \nu = \sin^2 \mu,$$

or since

$$\nu = 90^\circ - \varepsilon$$

that

$$\varepsilon = \pm \mu; \qquad \ldots (24)$$

thus the local isobar must be in a characteristic direction.

This/

This result was obtained differently in the more complete analysis of Craggs - Ref. 9.

Combining equation (23b) with the fact that

$$\tan \nu = \frac{\lambda\theta_s}{\lambda_s} \text{ (from equations (15) and (2))}$$

gives, when integrated along a streamline

$$\theta = \pm \omega + \text{constant},$$

i.e., simple wave flow results if Bickley's criterion is continuously realized along a streamline. We may note, in passing, that the Jacobian $J^1 \equiv (u_x v_y - u_y v_x)$, which would be zero for simple wave flow, can be written

$$J^1 = \frac{q^2 a^2}{u^2} \frac{v_y^2}{u^2 - a^2} \qquad \ldots (22a)$$

in the present notation. Hence the Jacobian is infinite when $u^2 = a^2$ if v_y is non-zero. The significance of the singularity in the Jacobian is evident from considerations in the hodograph plane.

3.1.2 Solutions of the flow equations in the hodograph plane are only acceptable if the transformation to the real plane is non-singular. This condition is satisfied if the Jacobian

$$j = \frac{\partial(x,y)}{\partial(u,v)} \left(\equiv \frac{1}{J^1} \right)$$

is non-zero. It is suggested in the literature that the occurrence $j = 0$ leads to the breakdown of potential flow. Lines along which $j = 0$ in the hodograph plane give rise to the so-called limit line in the flow plane. The properties of such a limit line are dealt with in Ref. 9, for example, and need not be considered in full herein. It is noted in passing that at a limit line the streamlines have cusps and the acceleration is infinite. This latter fact was shown in equation (22a) which thus provides the link between the limit line and Bickley's criterion for potential flow breakdown (see also Ref. 9).

Kármán in Ref. 12 was the first to give any geometrical significance to the vanishing of the transformation Jacobian. Kármán considered the Jacobian

$$j_1 = \frac{\partial(\phi,\psi)}{\partial(q,\theta)} = 0,$$

which reduces to $(1-M^2)\psi_\theta^2 + q^2\psi_q^2 = 0$ when the equations of motion in the hodograph plane are used. If δ is the angle between the constant velocity line and the streamline in the hodograph plane, then

tan δ/

$$\tan \delta = -\frac{\psi_\theta}{q\psi_q}$$

so that for $j_1 = 0$ we must have $\tan^2\delta \equiv \tan^2\mu$ and the streamline touches the characteristic in the hodograph plane (see fig. (5)).

The results of Sections 3.1.1 and 3.1.2 may be summarized as follows:

(a) Bickley's criterion implies infinite acceleration when the velocity component along the pressure gradient is sonic, unless $v_y = 0$.

(b) This condition of infinite acceleration indicates the formation of a limit line.

(c) Bickley's criterion is satisfied by a simple wave flow (but since $v_y = 0$ here, the acceleration is finite - except where the characteristics form an envelope). For a limited region of supersonic flow it was shown in Section 2.3 that simple wave flow was not possible. Hence Bickley's criterion always implies a flow breakdown somewhere in a limited supersonic region, although not necessarily at all points at which the criterion itself is satisfied.

3.1.3 Nikolski and Taganov in Ref. 8 developed a criterion for the breakdown of potential flow. The physical background to their method is that all outgoing, ξ, characteristics from the surface are to end on the sonic line (i.e., none end on a shock wave). The result of equation (17b) - giving $\tau_s \leqslant 0$ (or $T \geqslant 0$) - then gives a limitation to the velocity distribution on the surface which may be written as

$$\frac{d\lambda}{d(-\theta)} \leqslant -\lambda \tan\mu \quad \text{along a streamline,}$$

since from equations (16) and (16a):

$$\frac{\lambda_s}{\theta_s} = -\lambda \tan\mu \left(\frac{T - S}{T + S}\right)$$

and $T > 0$, $S > 0$. Hence the condition for flow breakdown taken in Ref. 8, is

$$\frac{d\lambda}{d(-\theta)} = -\lambda \tan\mu. \qquad \dots (25)$$

Comparison with the equations of Section 3.1.1 show this criterion to be identical with that of Bickley and thus deserves no further comment.

3.2/

3.2 Other considerations*

The previous section related the limit line formation and the Bickley, and Nikolski and Taganov criteria for potential flow breakdown. In essence the criteria demand that the acceleration at some point be infinite.

The criterion of the last paragraph (equation (25)) shows that if θ_s is finite then so will be λ_s (except possibly at $M = 1$). Hence the only possibility for the flow past a non-singular boundary to have infinite acceleration would be if this occurred on another streamline in the flow.

3.2.1 Several authors have considered the possibility of the formation of an envelope of characteristics in the flow field away from the surface. The conditions for envelope formation, expressed in terms of the rate of convergence of the characteristics, are less precise than the results discussed in Section 3.1 since they are expressed by inequalities of the type $\alpha_s \geqslant 0$, $\alpha_n \leqslant 0$.

To study these conditions we develop some aspects of the geometry of the characteristics in the local supersonic flow region, thus avoiding the confusion evident in the literature (see: e.g., Ref. 7). An important observation follows from equation (11a). We note that for the flow past a convex surface ($\theta_s < 0$):

(a) $\alpha_s = 0$ is only possible in a compressing flow;

(b) $\beta_s = 0$ is only possible in an expanding flow;

(c) in compressing flow $\beta_s < \alpha_s$, while in expanding flow $\alpha_s < \beta_s$.

Hence we conclude that for compressing flow, the η family of characteristics converge more rapidly than the ξ family, i.e., it is the incoming family of waves that will tend to form an envelope. For expanding flow the families of characteristics reverse their roles. This result indicates that any criterion for envelope formation of the ξ waves in compressing flow will be misleading (see Ref. 7 and 16).

From equations (10) and (11) it follows that for $\alpha_s = 0$ we have:

$$\frac{dM}{d\theta} = M\sqrt{M^2 - 1} \qquad \ldots (26)$$

along a streamline.

Similarly, for the η family of characteristics we find when $\beta_s = 0$:

$$\frac{dM}{d\theta} = -M\sqrt{M^2 - 1}. \qquad \ldots (26a)$$

For/

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*The analysis presented in this section was developed after discussions with Mr. G. Y. Nieuwland of the Nationaal Lucht- en Ruimtevaartlaboratorium, Amsterdam.

For limit line formation we have (Section 3.1) $\tan\varepsilon = \pm\tan\mu$, giving:

$$\frac{dM}{d\theta} = \pm\frac{M\left(1+\frac{\gamma-1}{2}M^2\right)}{\sqrt{M^2-1}} \qquad \text{... (26b)}$$

which is equivalent to $\frac{d\omega}{d\theta} = \pm 1$. That is, the condition for limit line formation and simple wave flow are given by the same expression.

The equations (26) and (26b) are presented in fig. (6). The following conclusions are evident from consideration of fig. (6):

(i) The Mach number gradient, $-\frac{dM}{d\theta}$, for limit line formation is a minimum at $M^2 = \frac{4}{3-\gamma}$ ($M = 1{\cdot}581$ for $\gamma = 1{\cdot}400$).

(ii) If a limit line forms at $M^2 = \frac{4}{3-\gamma}$, the equations (26), (26a) and (26b) show that either α_s or β_s is zero; see also equation (11c) which indicates that $\alpha_s = 0$ when $M^2 = \frac{4}{(3-\gamma)}$ and $T = 0$ (simple wave), and correspondingly that $\beta_s = 0$ when $M^2 = \frac{4}{(3-\gamma)}$ and $S = 0$ (simple wave).

Equation (11c) shows that α_s (and similar results hold for the η waves) can change sign with increasing Mach number for various values of θ_s and T. Fig. (6) includes the locus of conditions under which this change of sign takes place. This implies that a convergence of characteristics can result on either side of the streamline, depending upon the relation existing between the parameters. However, the other family of waves always forms a limit line first on the concave side of the streamline (since $\beta_s < \alpha_s$ for compressing flow). This result was obtained differently in Ref. 14 - see Section 3.2.2.

The above remarks are, to some extent, in contradiction to the suggestions of Laitone in Ref. 7 - particularly in connection with the formation of envelopes and in the significance of the Mach number $\sqrt{\frac{4}{3-\gamma}}$. This is due to the fact that in Ref. 7 the results are limited to the special case $\alpha_s = 0$ and confusion with generality follows.

Finally we collect together the following conditions holding at $M = \sqrt{\frac{4}{3-\gamma}}$:

(a) The characteristics have an inflexion point (which is not related to the inflexion in the Prandtl-Meyer function except for the special value 1·4 of γ).

(b) In general, the velocity and flow direction vary monotonically and continuously. At the special points where $\alpha_s = 0$ or $\beta_s = 0$ then this result need not hold - see Ref. 7.

(c)/

(c) The Mach number $\sqrt{\frac{4}{3-\gamma}}$ in no way represents a maximum obtainable local Mach number, nor need the cusp of a limit line occur at this Mach number; i.e., the critical velocity $\lambda = \lambda_{crit}$ in fig. (5) does not need to take the value $\sqrt{2}$.

3.2.2 Various attempts have been made to prove that infinite acceleration is mathematically impossible in a local supersonic flow region. We deal with two such cases.

Firstly, Nikolski and Taganov in Ref. 8 gave a long proof which attempts to show that if there is no singularity on the surface, (i.e., if λ_s and θ_s are finite) there cannot be one on any other streamline in the local supersonic flow region. Using the result of equation (25) we see that if θ_s is finite, λ_s will be also, along the same streamline (λ_n and θ_n must also be finite from equation (2)). Thus follows Nikolski and Taganov's first result - that λ_s etc. are only singular if θ_s is infinite. (This result may not be true at $M = 1$; see below.)

The final part of the proof of Ref. 8 takes several pages and will not be reproduced here. The essence of the proof, however, follows from the equations (16) and (16a) where the quantities λ_s, θ_s etc., are written in terms of the wave strengths T and S. Except at $M = 1$, λ_s and θ_s are only infinite if T and S are infinite. By differentiation along the characteristics, Nikolski and Taganov show that if S and T are finite on a bounding streamline, then they will be finite in the whole supersonic flow region. Hence the result follows. At $M = 1$ the above argument breaks down and Nikolski and Taganov conclude that if infinite acceleration does arise it does so at the sonic line. (This would certainly be in agreement with the result of Emmons in Ref. 13.)

This proof is correct and valid only if the conditions under which it was formulated hold. One condition is that two characteristics of the same family must not cross (the velocity field is single valued); hence the above result is only true if waves do not cross. The proof does not, however, eliminate the possibility of the formation of envelopes of characteristics with resulting infinite acceleration. For example, in the case of the supersonic flow in a concave bend, the equations of Section 2.1 are valid along a characteristic only up to the point where two waves cross; but this point is not determined by considering the variation of quantities along a single wave.

Mention should also be made of the work of Morawetz and Kolodner in Ref. 14, who, following Friedrichs[15], attempt to show that the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ cannot vanish for the type of flow under consideration. With the assumption that the derivatives $\psi_{\theta\theta}$, $\psi_{\theta q}$, ψ_{qq} exist and are bounded in the supersonic flow region of the hodograph plane they prove:

(a) A limit line cannot appear in a plane continuous flow past an aerofoil of finite curvature if the flow depends continuously on the freestream Mach number.

(b) For a set of flows which depend continuously on a parameter, a limit line will only form for some value of this parameter if the profile simultaneously has infinite curvature at some point.

This/

This result of Morawetz and Kolodner is significant in that it suggests that limit lines can only enter the flow field through the boundary streamline (solid surface). Hence if the boundary is uniform (θ_s finite) at all points the flow should not contain a singularity. However, the theory of Ref. 14 does invoke the theory of characteristics and it could be that the criticisms made above concerning the work of Nikolski and Taganov are also relevant here. Indeed Tsien[18] made an equivalent comment concerning the original work of Friedrichs. Manwell - (Ref. 20) - made similar deductions to those of Morawetz and Kolodner. These writers use quantities which are inversely proportional to the wave strengths of the present work.

In relation to this problem, mention may be made of Ref. 16 where Tollmein and Schäfer construct flow patterns about convex surfaces which contain envelopes of characteristics in the flow field. Certain approximations were, however, made in the theory of Ref. 16 and these could lead to doubts concerning the exactness of the flows obtained. In particular the comments made in Section 3.2.1 are relevant.

In Ref. 17 an attempt was made to use the criteria of limit line formation in practice. It was found that shock waves formed at Mach numbers well below that required for limit line formation.

4. Conclusions

The first part of the paper - Section 2 - obtained the following properties of a local supersonic flow region in steady, two-dimensional potential flow. In general the results are only valid in a region where the characteristics end on the sonic line.

(a) Waves incident on the sonic line must be expansion waves, while those leaving the sonic line must be compression waves.

(b) A characteristic cannot change from an expansion wave to a compression wave (or the reverse) and hence can have only one end on the sonic line.

(c) Along the expansion wave, the velocity and flow direction monotonically decrease towards the sonic line. Along the compression wave the velocity monotonically increases and the flow direction monotonically decreases away from the sonic line.

(d) Two waves which meet on the sonic line are of the same strength, while two which meet on the surface are of different strengths, the relative magnitudes depending on the sign of the pressure gradient.

(e) The isobar is at a smaller angle to the flow direction than is a characteristic.

(f) The rate of change of velocity along a streamline must be less than that required of simple-wave flow.

(g) Infinite acceleration on a streamline must be accompanied by infinite curvature of the streamline.

Certain/

Certain restrictions are to be imposed on the flow if it is to remain potential - Section 3. One restriction is that the velocity gradient must nowhere exceed that of simple-wave flow, since (as was shown in Section 2.3), simple-wave flow cannot exist up to the sonic line in a finite region. In a real flow there is also a restriction on velocity gradient if the boundary layer is to remain unseparated. This question was not considered herein but should always be borne in mind in any practical situation.

The other restriction is more obscure and demands that the streamline in the hodograph plane should not cross the Scherberg critical ellipse. When the streamline does cross this ellipse infinite acceleration results in the flow plane and limit lines form.

The result presented in Section 3.2 would indicate that a limit line can only enter the flow through the boundary streamline and not by the coalescence of characteristics in the supersonic region. This latter result follows only for convex ($\theta_s < 0$) surfaces. However the formation of a limit line demands an infinite curvature of the streamline and hence it remains a philosophical point as to what happens for increasing freestream Mach number in the flow about a given smooth surface; it may well be that the simple-wave flow limitation then governs the flow.

Finally, we note that in practice shock waves often form before the theoretical prediction of limit line formation, and so the consideration of the steady potential flow model of the local supersonic region in isolation seems inadequate.

Acknowledgement

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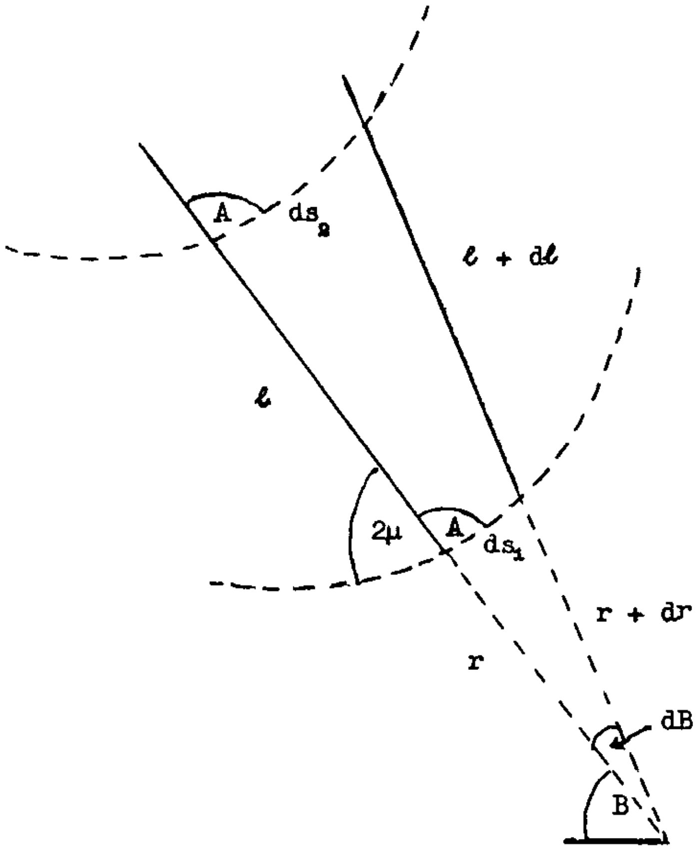
Appendix/

Appendix

Nikolski and Taganov's proof of the non-realization of simple wave flow in a finite supersonic flow region.

Assume simple wave flow between two members of the η family (say). The following flow diagram then holds:



Then $\qquad ds_1 \cdot \sin A = r\,dB ,$

and $\qquad ds_2 \quad \sin A = (r + \ell)\,dB ,$

but $\qquad d\ell = (ds_2 - ds_1) \cos A .$

Hence $\qquad \dfrac{d\ell}{\ell} = \cot A\, dB . \qquad \ldots (a)$

Noting that $\qquad \beta = \mu - \theta . \qquad \ldots (b)$

Since the flow is assumed to be simple wave compression we can put

$$\theta = \omega + \Theta \qquad \text{where } \Theta = \theta)_{\omega = 0}.$$

Then since

$$\omega = \mu + k \cot^{-1} (k \tan \mu) - \frac{\pi}{2} , \quad k^2 = \frac{\gamma+1}{\gamma-1}$$

[by equations (6a) and (5)]

we have

$$\beta = \frac{\pi}{2} - k \cot^{-1} (k \tan \mu) - \Theta$$

Then/

Appendix (cont'd)

Then putting

$$-\psi = \beta - \frac{\pi}{2} + \theta$$

gives

$$k \tan \mu = \cot \frac{\psi}{k}$$

Using

$$\cot 2\mu = \frac{1 - \tan^2 \mu}{2 \tan \mu}$$

gives

$$\cot 2\mu = \frac{k^2 - \cot^2 \psi/k}{2 k \cot \psi/k}$$

$$= \frac{1}{2}\left\{k \tan \frac{\psi}{k} - \frac{1}{k} \cot \frac{\psi}{k}\right\}$$

so that equation (a) may be integrated to give

$$\ell n\, \ell = -\frac{1}{2}\left\{ k^2\, \ell n \cos \frac{\psi}{k} + \ell n \sin \frac{\psi}{k} \right\} + \text{const}^*.$$

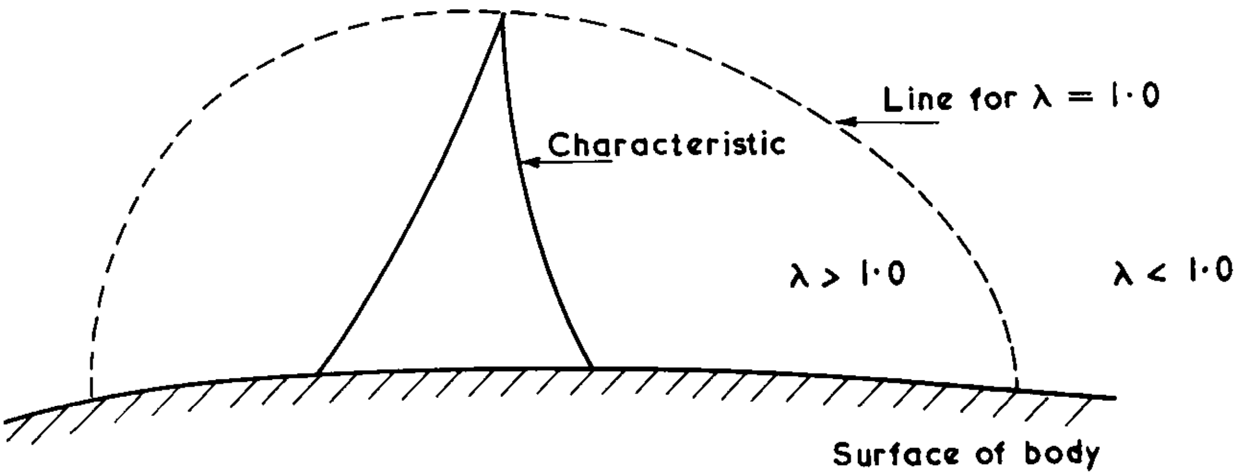
Hence

$$\ell n \left\{\ell^2 \left(\cos \frac{\psi}{k}\right)^{k^2} \sin \frac{\psi}{k}\right\} = \text{const};$$

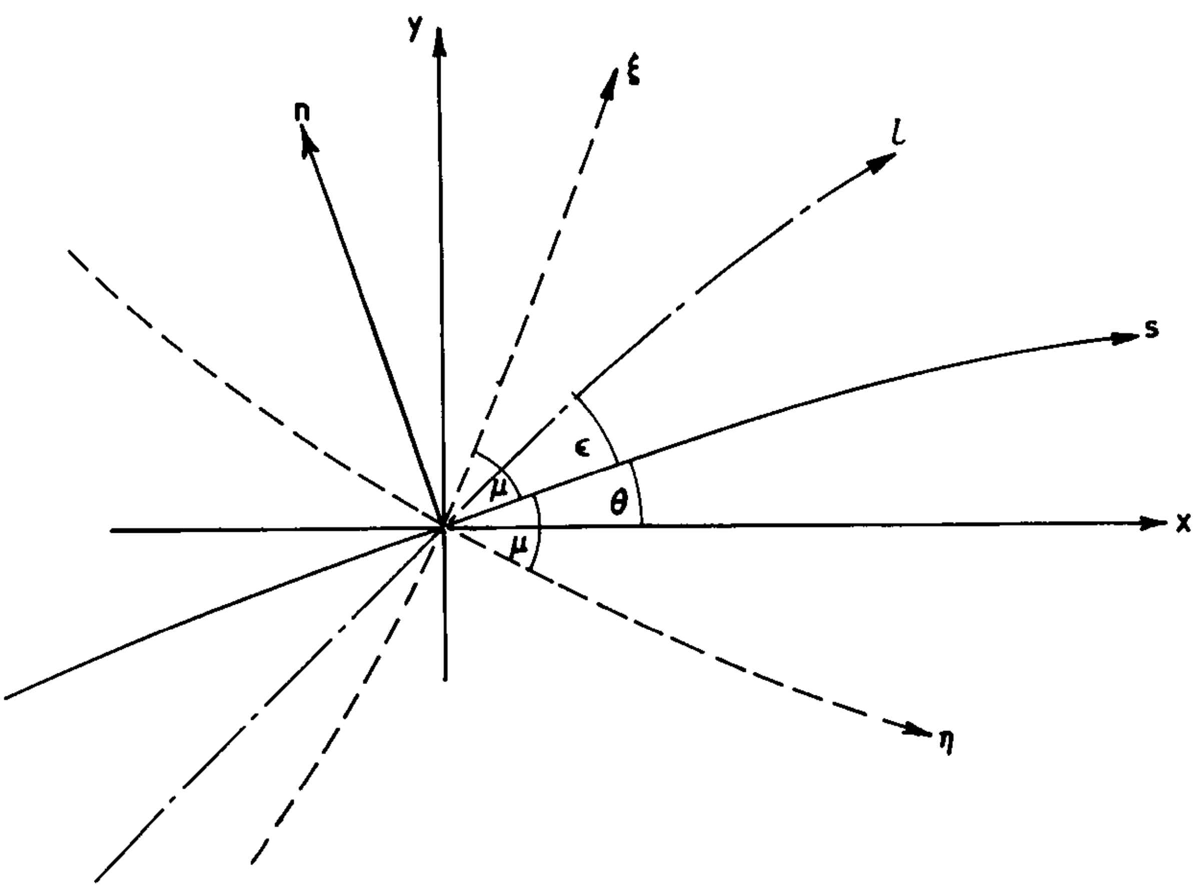
since sonic conditions correspond to $\psi = 0$ then $\ell \to \infty$ as sonic conditions are reached.

DS
HD

* Where 'ℓn' denotes the natural logarithm.

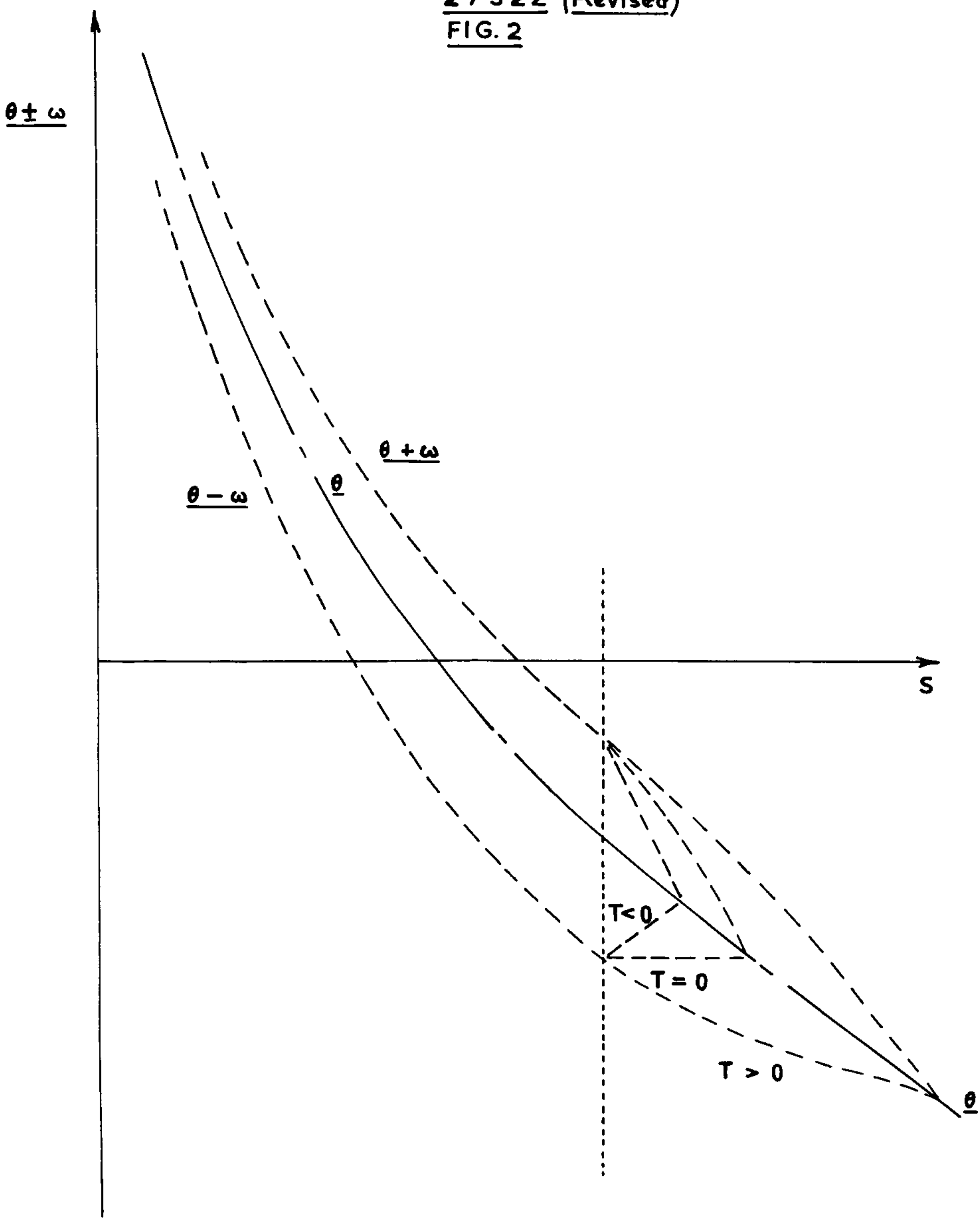


(a) A local supersonic flow region



(b) Coordinate system

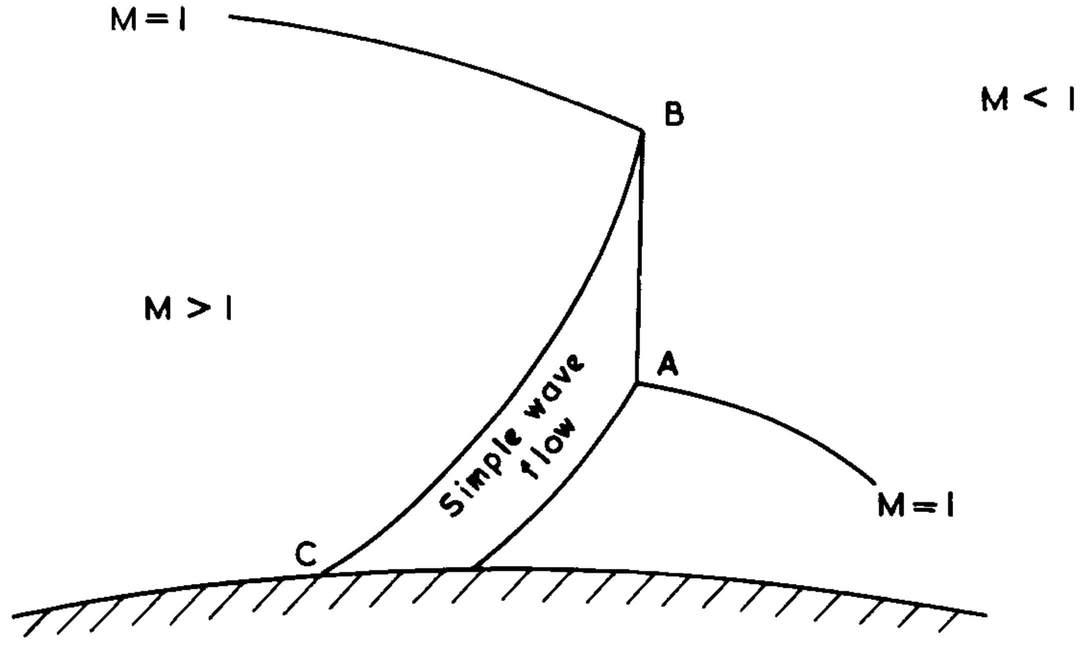
27322 (Revised)
FIG. 2

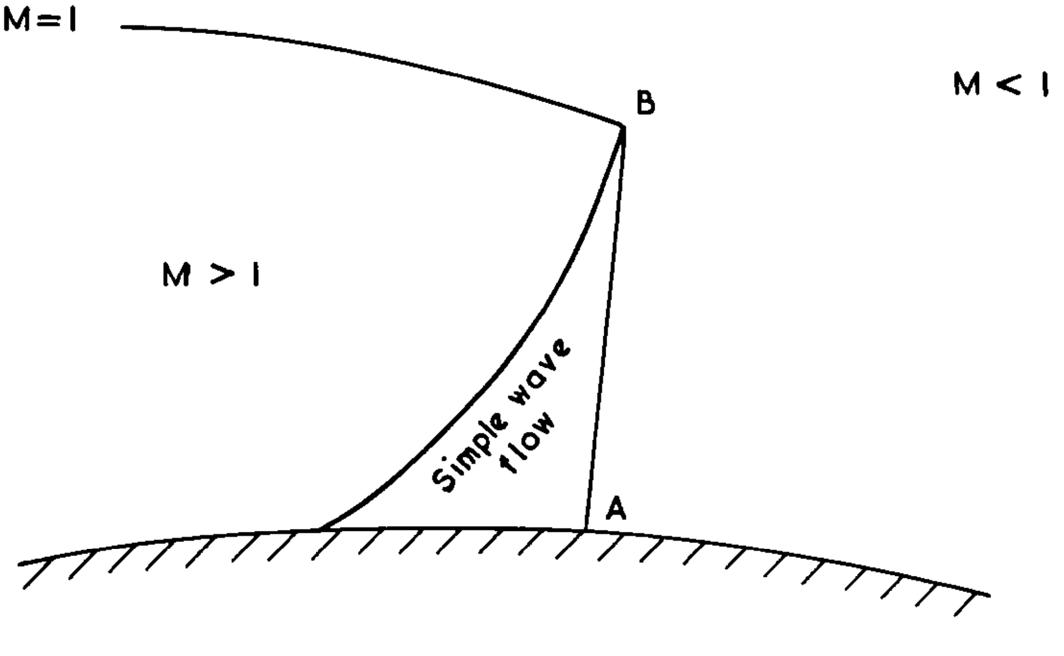


Significance of wave strength in compressing flow

27322 (Revised)

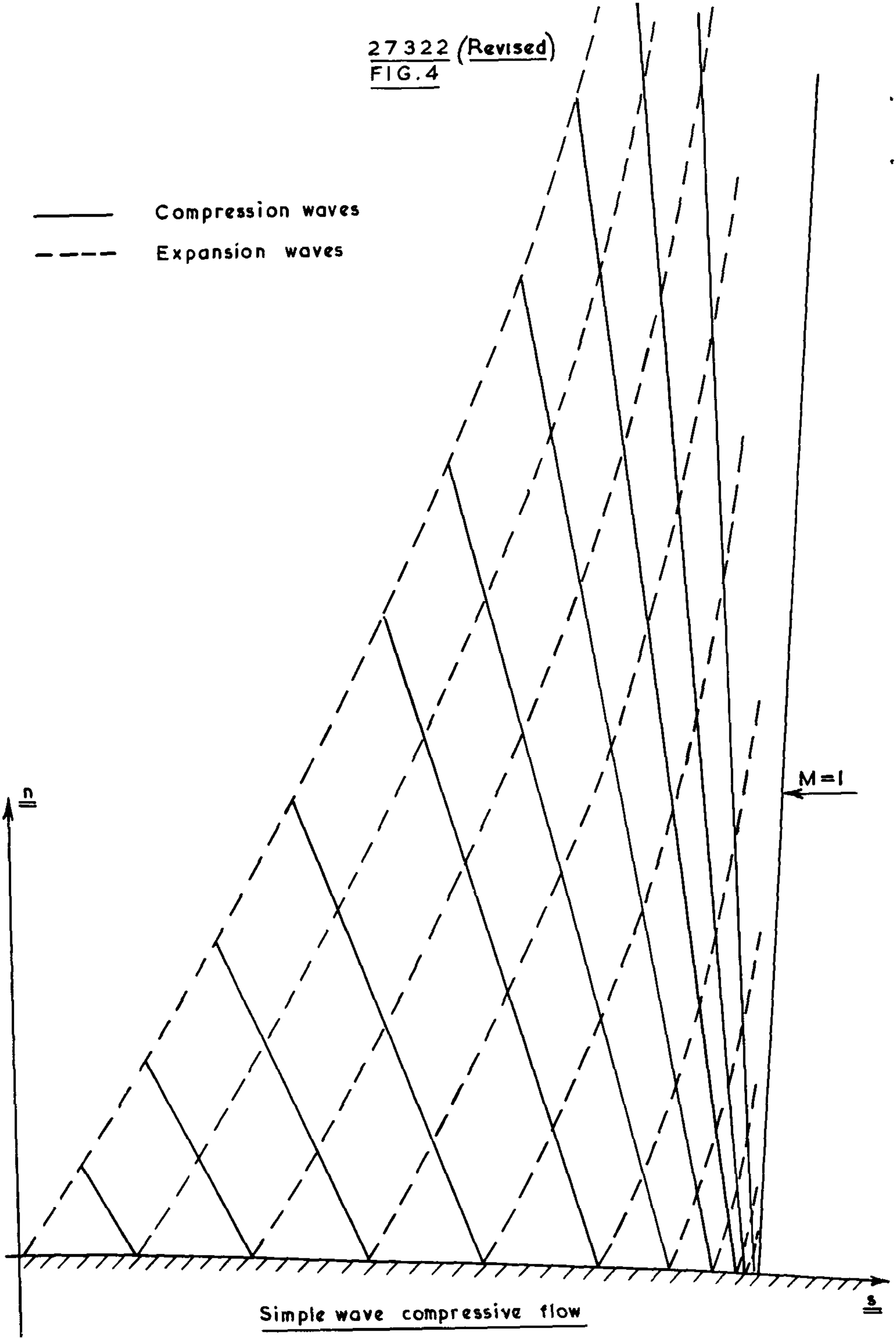
FIG 3





Types of regional simple wave flow along η characteristics

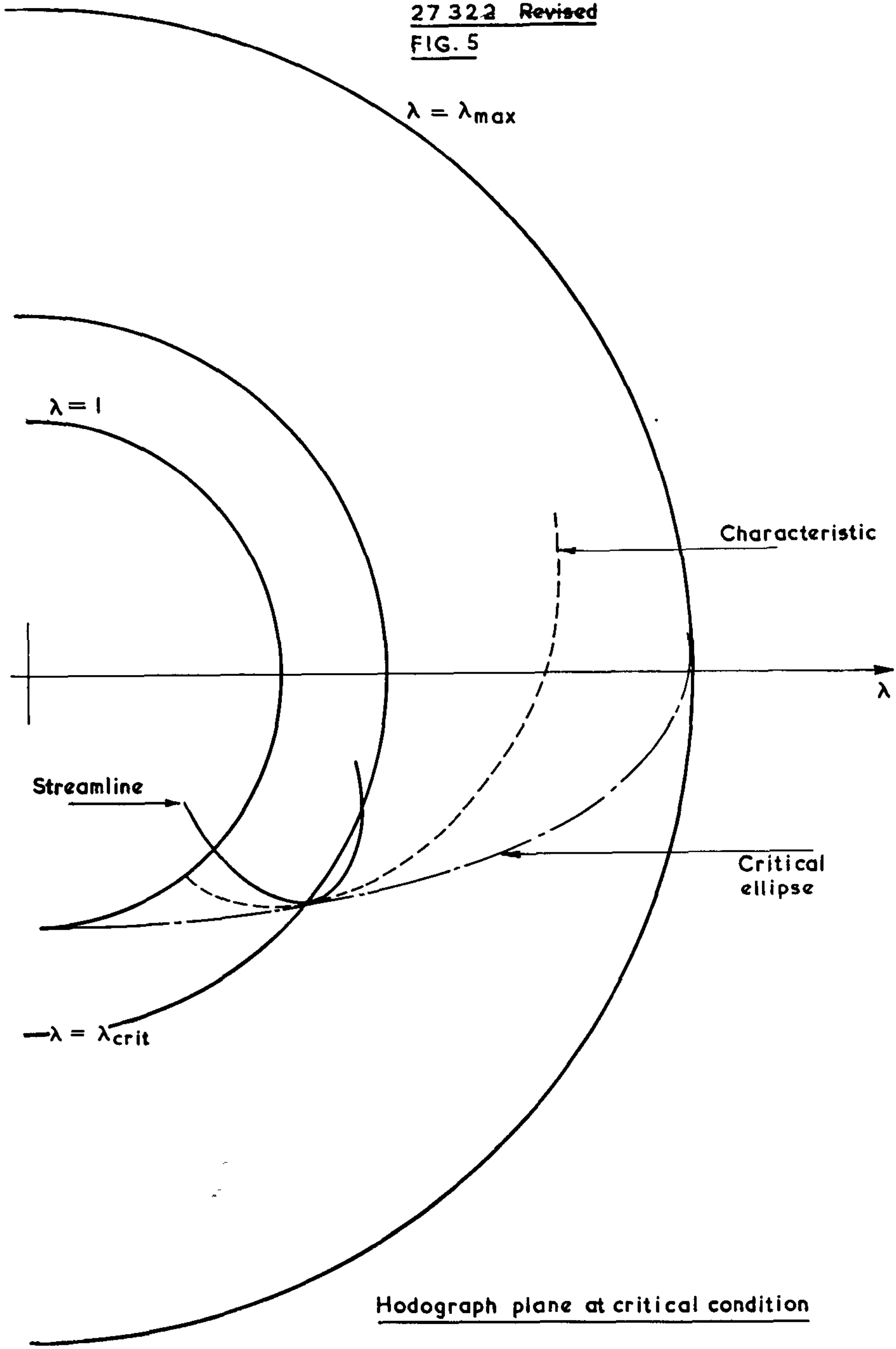
27322 (Revised)
FIG. 4



Simple wave compressive flow

27 322 ~~Revised~~

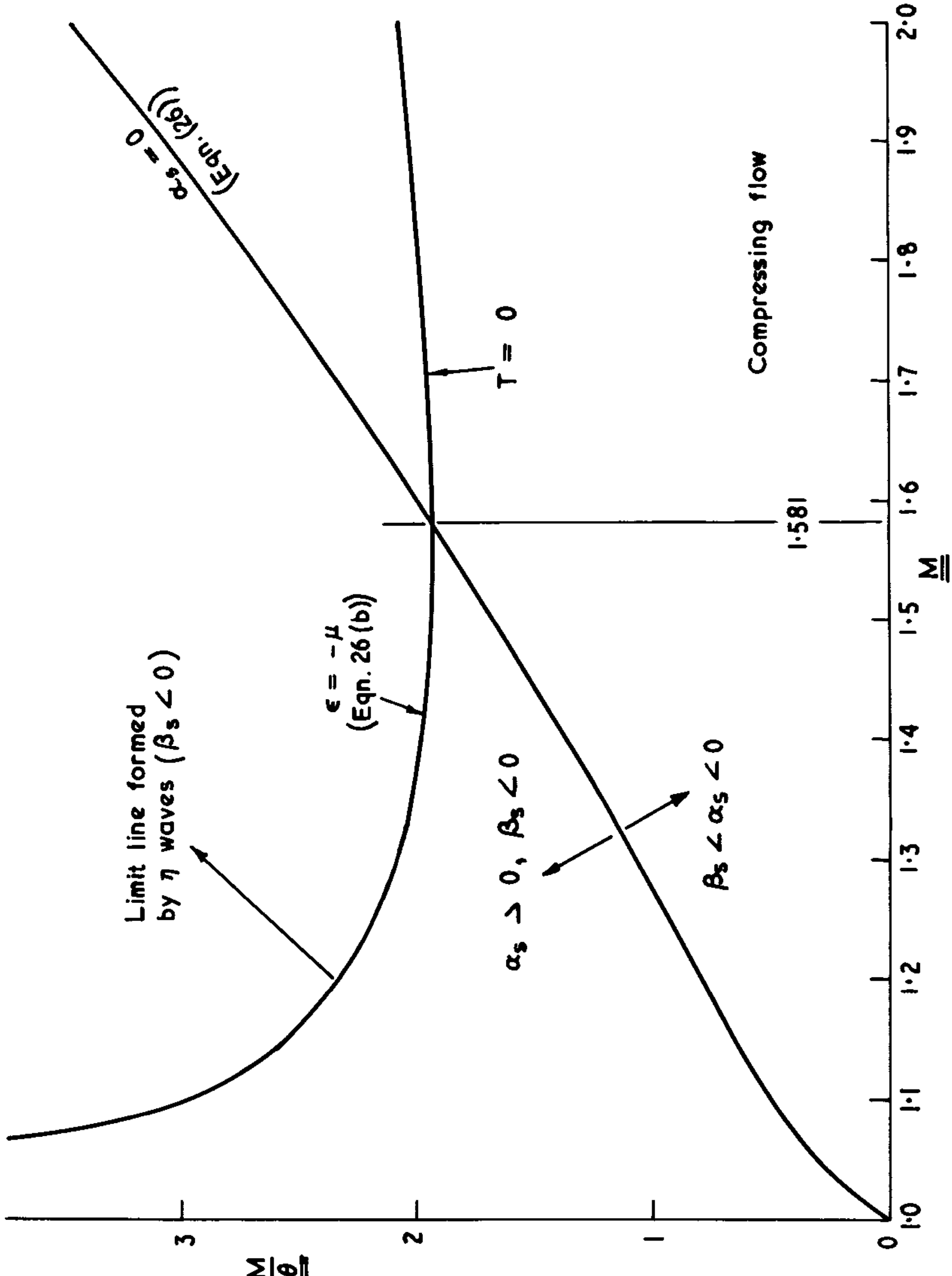
FIG. 5



Hodograph plane at critical condition

27 322 (Revised)

FIG. 6



Conditions of flow with $\theta_s < 0$ (convex surface) $\gamma = 1{\cdot}400$

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January 1967
Moulden, T. H.

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