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A MARCHING PROCEDURE FOR THE DETERMINATION OF INVISCID TWO-DIMENSIONAL SONIC FLOW PAST A BLUNT SYMMETRICAL BODY

by

D. G. Randall

SUMMARY

The equations of motion for two-dimensional, inviscid, sonic flow are written in a form that permits the introduction of a marching procedure for determining the flow past a symmetrical, blunt body. The independent variables are transformed to new variables, θ and s, such that points infinitely far from the body are mapped into the line $\theta = 0$, the line of symmetry is mapped into the line $s = 0$, and the body contour is mapped into the line $\theta = 1$; marching takes place in the s direction, starting from the line $s = 0$. The dependent variables are also transformed, in such a way that the physical quantities have the correct asymptotic behaviour far from the body.

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solution for high subsonic flow past a curved aerofoil, although the numerical work was heavy. Oswatitsch and Keune[7] gave reasons for believing that for sonic speed ($M_\infty = 1$) the factor Φ_{XX} on the right-hand side of equation (1.1) could be approximated by a constant; this approximation linearises the potential equation, turning it into the equation of heat conduction. This approach was exploited and extended by several authors[8,9,10]. Cole and Royce[11] argued that turning equation (1.1) into a parabolic equation was unsatisfactory for physical reasons and that a better approach was to replace Φ_X by a linear function of X; this approximation makes the equation elliptic on one side of a plane normal to the X axis and hyperbolic on the other side. Cole and Royce applied this technique to the axi-symmetric problem; later, Evans[12] extended it to the two-dimensional problem. Although replacing Φ_X or Φ_{XX} by at most a linear function of X involves drastic simplifications, the results obtained by these methods are in surprisingly good agreement with experiment.

The most recent developments in the theory have been further investigations into the behaviour of sonic flow at large distances from the body. A number of authors[13,14,15] have independently obtained a representation in a simple closed form of the asymptotic behaviour of Φ in axi-symmetric flow; this was previously available in numerical form only[3]. The representation is achieved by changing from X, Y (Y now being the radial coordinate) to new independent variables σ, τ; these new variables are such that small values of τ correspond to large distances from the body. It is possible to expand Φ in powers of τ, the coefficients being functions of σ; the first term in the expansions is the representation obtained numerically in Ref. 3. Further terms in the expansion are derived in Ref. 15, where it is shown that the powers of τ are not, in general, integral; nevertheless, the functions of σ forming the coefficients of the powers of τ can all be obtained in simple closed form. The most far-reaching advances have been made by Euvrard[16], who has obtained a similar expansion for the axi-symmetric problem valid when the governing equations are the full inviscid equations of motion. He indicates that all the functions of σ can still be obtained in simple closed form; as before, the powers of τ with which they are associated are not, in general, integral. Euvrard has also obtained[17] a corresponding expansion for the two-dimensional sonic problem, again based on the full inviscid equations. The functions of σ can again be obtained in simple closed form, but this time the powers of τ with which they are associated are all integral.

1.2 Marching procedures

The present paper describes a method for determining the steady, sonic flow past a symmetrical, blunt, two-dimensional body, when the governing equations are the exact inviscid equations of motion. It is shown that a knowledge of Euvrard's results for the asymptotic behaviour of the flow allows the equations of motion to be solved by a method similar to that employed by Mangler and Evans[18] in their work on hypersonic flow past a symmetrical, blunt, two-dimensional body. Mangler and Evans argue that Ψ, the stream function, can be chosen to be zero along the line of symmetry and on the body and that it is then equal to ρ_∞ U Y on the bow shock, where ρ_∞ and U are respectively the free-stream density and speed. By transforming the independent variables from X, Y to Σ, Y, where $\Sigma = \Psi/(\rho_\infty U Y)$, the upper half of the physical plane is mapped on to a semi-infinite strip bounded by the lines $\Sigma = 1$ (corresponding to the shock), $\Sigma = 0$ (the body), and $Y = 0$ (the line of symmetry). If the dependent variables are known along a line of constant Σ, their derivatives with respect to Y can be found numerically; the equations of motion can then be used to obtain the derivatives with respect to Σ; this allows an approximate calculation of the dependent variables along a line of constant Σ lying at a sufficiently small distance from the first one. Once the shock shape is prescribed, all the dependent variables can be found along the line $\Sigma = 1$, so that the marching procedure can be started from there. The marching ends when the line $\Sigma = 0$ is reached. The success of the procedure depends upon the fact that Ψ/Y tends to a finite limit as Y tends to zero; for this to be true the X axis must be a line of symmetry. For the transonic problem it is also necessary for the governing equations to be the full inviscid equations (and not the transonic approximation to these) and for the body to be blunt. The reason for both these requirements is that an involved singularity exists at the nose if either of them is relaxed.

There are two significant differences between the marching procedure employed by Mangler and Evans and the one employed here. First, whereas the line $\Sigma = 1$ corresponds in hypersonic flow to the bow shock, in sonic flow it corresponds to points infinitely far from the body. Now, in hypersonic flow the dependent variables along the line $\Sigma = 1$ are undetermined to the extent of an arbitrary function (the shock shape); on the reasonable assumption that each choice of shock shape leads to a different body, it is clear that, in theory, an enormous variety of bodies can be obtained; of course, the very nature of the marching procedure precludes any possibility of prescribing the body shape beforehand. In contrast, in sonic flow the dependent variables along the line $\Sigma = 1$ are undetermined only to the extent of an arbitrary constant; this is because, apart from a scaling factor, all two-dimensional bodies have the same asymptotic behaviour in sonic flow[3]. For this reason, the marching procedure for the sonic problem takes place effectively in the Y direction. It can be shown that along the line $Y = 0$ appropriately chosen dependent variables are undetermined to the extent of an arbitrary function; the situation is then the same as for the hypersonic problem. Making a special choice of the arbitrary function is here equivalent to prescribing the pressure distribution along the line of symmetry from free-stream conditions to the stagnation point at the nose.

A further difference is that for the hypersonic problem a marching procedure is possible for both two-dimensional and axi-symmetric flow. For the sonic problem an extension to axi-symmetric flow cannot be made; this is connected with the fact mentioned earlier that the powers of τ in the asymptotic expansions of the dependent variables are integral in two-dimensional flow but not in axi-symmetric flow (see Section 2.3 for further details). Other differences between the marching procedures in the hypersonic and the sonic problems are ones of detail only.

1.3 Purpose of present investigation

Each choice of the arbitrary function determining the behaviour of the dependent variables along the line $Y = 0$ presumably leads to a different body. The ultimate aim of the present investigation is to produce bodies whose maximum thicknesses are small compared with their lengths and to find the pressure distributions over them. The distributions can then be compared with the results obtained by approximations in current use[7,8,9,10,11,12], since these are intended to be applied to the type of body mentioned above. By this means it is hoped to decide which of the currently used approximations is the most accurate. Admittedly, these approximations have usually been applied to determine pressure distributions on bodies with pointed noses; nevertheless, at least one of them[9] has been used on a round-nosed body, and it may well be possible to make this extension to some of the other approximate theories. In any case, it should be possible to make a comparison between results obtained by the present method and results obtained by approximate theories applied to a body having a pointed nose that fairs into the original body a short distance downstream of the nose.

The present paper carries the investigation to an intermediate stage only. In Section 2 new dependent and independent variables are introduced that put the equations of motion in a form suitable for a marching procedure. It is shown that the curve along which the velocity component in the Y direction vanishes is a singular line in the transformation from old to new coordinates, so that the marching procedure cannot give results beyond this line without modification. There seems to be no insuperable obstacle to the introduction of such a modification, and it is hoped to attempt this at a later date. Since this is only an interim report, the analysis in Section 2 has been considerably condensed.

Section 3 contains a description of a Mercury Autocode program for carrying out the marching procedure. The program has been used to calculate one example; the results obtained suggest that the marching procedure is stable.

2 DERIVATION OF EQUATIONS

In sonic flow the free-stream speed is equal to the speed of sound, a_∞. Let R_o be some representative length; for example, R_o could be set equal to the radius of curvature of the body at the nose. Introduce non-dimensional quantities x, y, p, ρ, u, v, ψ and ϕ, such that

R_o x, R_o y are rectangular cartesian coordinates (y = 0 being the line of symmetry),

$\rho_\infty a_\infty^2 p$ is the pressure,

$\rho_\infty \rho$ is the density,

$a_\infty u$, $a_\infty v$ are the velocity components in the x and y directions respectively,

$\rho_\infty a_\infty R_o \psi$ is the stream function (ψ is chosen to be zero on the body and along the line of symmetry), and

$a_\infty R_o \phi$ is the velocity potential.

There are four equations for the dependent variables p, ρ, u and v in terms of the independent variables x and y: the continuity equation; an equation expressing the fact that the flow is irrotational; Bernoulli's equation; an equation expressing the fact that the flow is homentropic. The boundary conditions associated with these equations are that on the body the normal velocity component must be zero and that at large distances from the body the dependent variables must tend to their free-stream values in a prescribed manner[3]; the free-stream values are given by $\rho = 1$, $u = 1$, $v = 0$, and $p = 1/\gamma$.

The four equations mentioned above may be written as follows[3]:

$$\rho\, u_x + \rho\, v_y + u\, \rho_x + v\, \rho_y = 0 \quad , \tag{2.1}$$

the continuity equation;

$$u_y - v_x = 0 \quad , \tag{2.2}$$

the irrotational equation;

$$u^2 + v^2 + \frac{2\gamma}{(\gamma - 1)} \frac{p}{\rho} = \frac{(\gamma + 1)}{(\gamma - 1)} \quad , \tag{2.3}$$

Bernoulli's equation; and

$$\frac{\gamma p}{\rho^\gamma} = 1 \quad , \tag{2.4}$$

the homentropic equation. It is known[3] that there exists a shock starting from some point on the body; downstream of the shock equations (2.2) and (2.4) are not valid.

The non-dimensional stream-function, ψ, satisfies

$$\psi_y = \rho u \quad , \tag{2.5a}$$

$$\psi_x = -\rho v \quad ; \tag{2.5b}$$

elimination of ψ from equations (2.5) by cross-differentiation leads to equation (2.1). The non-dimensional velocity potential, ϕ, satisfies

$$\phi_x = u \quad , \tag{2.6a}$$

$$\phi_y = v \quad ; \tag{2.6b}$$

elimination of ϕ by cross-differentiation leads to equation (2.2).

From equations (2.3) and (2.4),

$$\rho = [\tfrac{1}{2}(\gamma + 1) - \tfrac{1}{2}(\gamma - 1)(u^2 + v^2)]^{1/(\gamma - 1)} \quad ; \qquad (2.7)$$

this can be used to turn equations (2.1) and (2.2) into equations having u and v as the only dependent variables.

2.1 The flow at large distances from the body

To obtain an expansion valid far from the body it is expedient[17] to replace the independent variables x, y by two quantities σ, τ, where

$$x = -(\gamma + 1)^{1/3}\,\mu(1 - 2\sigma)/\tau^2 \quad , \qquad (2.8a)$$

$$y = \sigma^{\frac{1}{2}}/\tau^{5/2} \quad . \qquad (2.8b)$$

Here, μ is an unspecified constant; it is effectively a scaling factor. By combining equations (2.6), (2.7) and (2.1) a second order partial differential equation for ϕ can be obtained; this can be used to determine an expansion for ϕ in powers of τ, the coefficients being functions of σ. It is found that

$$\phi = \frac{-(\gamma + 1)^{1/3}\,\mu(1 - 2\sigma)}{\tau^2} + \frac{4\,\mu^3(6 - 3\sigma + 2\sigma^2)}{3\tau} + \frac{\mu^5\, g_1(\sigma)}{(\gamma + 1)^{1/3}} + \ldots , \qquad (2.9)$$

where the first term is simply equal to x, the second term is the well-known[3] dominant term in the asymptotic expansion of the disturbance velocity potential, and the as yet undetermined function $g_1(\sigma)$ must not be singular when $\sigma = 0$. From equations (2.6) and (2.9),

$$u = 1 - \frac{4\,\mu^2(1 - \sigma)\tau}{(\gamma + 1)^{1/3}} + \frac{5\,\mu^4\,\sigma\, g_1'(\sigma)\tau^2}{2(\gamma + 1)^{2/3}\,(1 + 3\sigma)} + \ldots \quad , \qquad (2.10a)$$

$$v = \frac{8\,\mu^3\,\sigma^{\frac{1}{2}}(3 - 2\sigma)\tau^{3/2}}{3} + \frac{2\,\mu^5\,\sigma^{\frac{1}{2}}(1 - 2\sigma)\, g_1'(\sigma)\tau^{5/2}}{(\gamma + 1)^{1/3}\,(1 + 3\sigma)} + \ldots \quad , \qquad (2.10b)$$

where dashes denote differentiation with respect to σ. On substituting equations (2.10) into equation (2.2) it is found that g_1 satisfies the following ordinary differential equation;

$$\sigma(4-3\sigma)g_1'' + 2(1-2\sigma)g_1' = -\frac{16}{3}[3(2\gamma-1) - 24(2\gamma+1)\sigma + 13(6\gamma+5)\sigma^2 - 6(6\gamma+5)\sigma^3] .$$

The appropriate solution of this can be obtained in closed form; it is

$$g_1(\sigma) = \text{const} - \frac{8}{15}[15(2\gamma-1)\sigma - 5(6\gamma+5)\sigma^2 + 2(6\gamma+5)\sigma^3] . \qquad (2.11)$$

Equations (2.10) and (2.11) lead to

$$u = 1 - \frac{4\mu^2(1-\sigma)\tau}{(\gamma+1)^{1/3}} - \frac{4\mu^4\sigma[15(2\gamma-1) - 10(6\gamma+5)\sigma + 6(6\gamma+5)\sigma^2]\tau^2}{3(\gamma+1)^{2/3}(1+3\sigma)} + \ldots , \qquad (2.12a)$$

$$v = \frac{8\mu^3\sigma^{\frac{1}{2}}(3-2\sigma)\tau^{3/2}}{3} - \frac{16\mu^5\sigma^{\frac{1}{2}}(1-2\sigma)[15(2\gamma-1) - 10(6\gamma+5)\sigma + 6(6\gamma+5)\sigma^2]\tau^{5/2}}{15(\gamma+1)^{1/3}(1+3\sigma)} + \ldots . \qquad (2.12b)$$

The expansion for ψ is obtained by the following procedure. First, the expansion for ρ is obtained from equations (2.7) and (2.12); this expansion and equations (2.8) and (2.12) are then used to turn equations (2.5) into two linear simultaneous equations for ψ_σ and ψ_τ; on solving these and integrating it is found that

$$\psi = \frac{\sigma^{\frac{1}{2}}}{\tau^{5/2}} - \frac{8(\gamma+1)^{1/3}\mu^4\sigma^{\frac{1}{2}}(3-\sigma)(1+\sigma)}{3\tau^{\frac{1}{2}}} + \frac{16\mu^6\sigma^{\frac{1}{2}}[30(2\gamma-3) - 45(2\gamma+1)\sigma + 24(3\gamma+5)\sigma^2 - 8(3\gamma+5)\sigma^3]\tau^{\frac{1}{2}}}{45} + \ldots , \qquad (2.13)$$

where the constant of integration has been chosen so that $\psi = 0$ when $y = 0$; the first term in the expansion is simply equal to y. ψ could have been determined directly by writing it as an expansion in powers of τ of the form of equation (2.13) and then deriving ordinary differential equations for the unknown coefficients by using equation (2.2); but, since the expansion for ϕ is already available, it is simpler to use the above procedure.

2.2 Equations for the marching procedure

The first transformation is very similar to one employed in Ref. 18. The independent variables x, y are replaced by ζ, y, where

$$\zeta = 1 - \frac{\psi}{y} \quad . \qquad (2.14)$$

This transformation maps regions infinitely far from the body into the line $\zeta = 0$, the x axis into the part of the line $y = 0$ running from $\zeta = 0$ to $\zeta = 1$, and the body contour into the line $\zeta = 1$. The fact that the line $\zeta = 0$ corresponds to regions infinitely far from the body makes it very likely that the dependent variables will have singularities along this line. This is the reason for taking the new variable ζ to be $1 - \psi/y$ rather than ψ/y (as in Ref. 18); expansions in powers of ζ use less print than expansions in powers of $(1-\zeta)$.

The results collected in Section 2.1 are now used to investigate the behaviour of the dependent variables along the line $\zeta = 0$. From equations (2.13), (2.8b) and (2.14),

$$\zeta = \frac{8(\gamma+1)^{1/3}\mu^4(3-\sigma)(1+\sigma)\tau^2}{3} - \frac{16\mu^6}{45} \times$$

$$[30(2\gamma-3) - 45(2\gamma+1)\sigma + 24(3\gamma+5)\sigma^2 - 8(3\gamma+5)\sigma^3]\tau^3 + \ldots \quad . \qquad (2.15)$$

Equations (2.15) and (2.8b) give the relationship between ζ, y and σ, τ. It is known[15] that negative values of σ do not correspond to any part of the physical plane and that a shock intervenes before the point $\sigma = 3$, $\tau = 0$ is reached; hence, the coefficient of τ^2 in equation (2.15) is always positive. From equations (2.10), the expansion for u contains powers of $\zeta^{\frac{1}{2}}$ and that for v contains powers of $\zeta^{\frac{1}{4}}$. Consequently, at first sight it seems reasonable to introduce a new variable, θ, by writing $\zeta = \theta^4$; but in the equations that are shortly to be obtained v always appears in the form v^2 (this is connected with the symmetry of the flow), so that

$$\zeta = \theta^2 \qquad (2.16)$$

is a more appropriate change of variable. It also turns out that y always appears in the combination $y^2\theta^5$, where the appearance of y^2 rather than y is again connected with the symmetry of the flow. The factor θ^5 arises for the following reasons. From equations (2.8b), $\sigma = \tau^5 y^2$; from equations (2.15) and (2.16), τ behaves initially like θ; hence, σ behaves like $\theta^5 y^2$; and previous work[17] has shown that σ is an important variable. All this leads to the following transformation of independent variables;

$$\theta = \zeta^{\frac{1}{2}} = \left(1 - \frac{\psi}{y}\right)^{\frac{1}{2}} \quad , \tag{2.17a}$$

$$s = \theta^5 y^2 = \left(1 - \frac{\psi}{y}\right)^{5/2} y^2 \quad . \tag{2.17b}$$

Regions infinitely far from the body correspond to the line $\theta = 0$; the line $y = 0$ corresponds to the part of the line $s = 0$ running from $\theta = 0$ to $\theta = 1$; the body contour corresponds to the line $\theta = 1$. The partial derivative operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ can be obtained in terms of $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \sigma}$ by using equations (2.17) and (2.5).

Equations (2.12), the analogous expansions for p and ρ, and a considerable amount of hindsight suggest the following replacement of the dependent variables u, v, p, ρ by b_1, e_1, f_1 and h_1, where

$$u = 1 - \theta\, f_o(\theta) - \theta\, s\, f_1(\theta, s) \quad , \tag{2.18a}$$

$$v^2 = \theta^3\, s[b_o(\theta) + s\, b_1(\theta, s)] \quad , \tag{2.18b}$$

$$\frac{1}{\rho} = 1 - \theta\, e_o(\theta) - \theta\, s[f_1(\theta, s) - \theta\, e_1(\theta, s)] \quad , \tag{2.18c}$$

$$\gamma p = 1 + \theta\, h_o(\theta) + \theta\, s\, h_1(\theta, s) \quad . \tag{2.18d}$$

Here, $e_o(\theta)$, $f_o(\theta)$ and $h_o(\theta)$ are universal functions of θ (for any given value of γ), so that they can be calculated once and for all; $b_o(\theta)$ is largely arbitrary (see below for details). $f_o(\theta)$ satisfies the following equation;

$$(1 - \theta\, f_o)[1 + (\gamma - 1)\theta\, f_o - \tfrac{1}{2}(\gamma - 1)\, \theta^2\, f_o^2]^{1/(\gamma - 1)} = (1 - \theta^2) \quad . \tag{2.19}$$

$e_o(\theta)$ and $h_o(\theta)$ are given in terms of $f_o(\theta)$ by

$$e_o = \frac{(f_o - \theta)}{(1 - \theta^2)} \quad , \tag{2.20a}$$

$$h_o = \frac{e_o + (\gamma - 1)(2 - \theta\, f_o)\, f_o}{2(1 - \theta\, e_o)} \quad . \tag{2.20b}$$

Equations (2.17) and (2.18) lead to

$$[4(1-\theta\ e_o)+s(5-\theta^2)e_1+\theta s f_1]s b_{1s}$$
$$-2(5-\theta^2)(b_o+s\ b_1)s\ e_{1s}-2\theta(b_o+s\ b_1)s\ f_{1s}$$
$$=\ -\theta s[(1-\theta^2)e_1+\theta\ f_1]b_{1\theta}+2\theta(1-\theta^2)(b_o+s\ b_1)e_{1\theta}+2\theta^2(b_o+s\ b_1)\ f_{1\theta}$$
$$-[4(1-\theta\ e_o)-(1-\theta^2)s\ e_1+\theta s f_1]b_1+[2(3-\theta^2)b_o-\theta(1-\theta^2)b_o']e_1-\theta^2\ b_o'\ f_1 \qquad \ldots\ (2.21a)$$

(the continuity equation);

$$5\ s^2\ b_{1s}+2[4(1-\theta\ e_o)+s(5-\theta^2)e_1+\theta s f_1]s\ f_{1s}$$
$$=\ -\theta s\ b_{1\theta}-2\theta s[(1-\theta^2)e_1+\theta\ f_1]f_{1\theta}-(8b_o+\theta b_o')-13 s b_1$$
$$-2(1-\theta^2)(f_o+\theta f_o')e_1-2\{[4(1-\theta\ e_o)+\theta(f_o+\theta f_o')]+2(3-\theta^2)se_1+2\theta s f_1\}f_1 \qquad \ldots\ (2.21b)$$

(the irrotational equation);

$$[(1-\theta\ e_o)-\theta s\ (f_1-\theta\ e_1)]s\ h_{1s}-\gamma(1+\theta\ h_o+\theta s\ h_1)s\ (f_{1s}-\theta\ e_{1s})$$
$$=\ [\gamma(1+\theta\ h_o)+(\gamma+1)\theta s\ h_1]\ (f_1-\theta\ e_1)-(1-\theta\ e_o)h_1 \qquad (2.21c)$$

(the homentropic equation, differentiated with respect to s); and

$$\theta\ s^2\ b_{1s}-2(1+\theta\ h_o+\theta s\ h_1)s\ e_{1s}+2(f_o+h_o+s\ f_1+s\ h_1)s\ f_{1s}$$
$$=\ -\theta\ b_o-2\theta s\ b_1+2(1+\theta\ h_o+\theta s\ h_1)e_1-2(f_o+h_o+s\ f_1+s\ h_1)f_1 \qquad (2.21d)$$

(an equation obtained by differentiating the homentropic equation and Bernoulli's equation with respect to s and then combining the resulting equations so as to eliminate the term in h_{1s}). Dashes now denote differentiation with respect to θ.

Equations (2.21a), (2.21b) and (2.21d) are three equations for b_{1s}, e_{1s} and f_{1s} in terms of b_1, e_1, f_1, h_1, $b_{1\theta}$, $e_{1\theta}$, $f_{1\theta}$ and the dependent variables, θ and s. The determinant, D, of the left-hand side of these equations can be written in the form

$$D = -\frac{4\,s^3}{\theta^4}\left\{\gamma p\left[\frac{(5-\theta^2)}{\rho} - 5u\right]^2 + \left[25\gamma p - \frac{(5-\theta^2)^2}{\rho}\right]v^2\right\} , \quad (2.22)$$

where equations (2.18) have been used. Provided that $D \neq 0$, equations (2.21) can be solved for b_{1s}, e_{1s}, f_{1s} and h_{1s}, because, from equation (2.18c), the coefficient of b_{1s} in equation (2.21c) is simply $1/\rho$. In subsonic regions D is never positive, since

$$25\gamma p - \frac{(5-\theta^2)^2}{\rho} > 25\left(\gamma p - \frac{1}{\rho}\right) = 25\left(\rho\, a^2 - \frac{1}{\rho}\right)$$

$$= 25\rho\left(a^2 - \frac{1}{\rho^2}\right) ,$$

where $a_\infty a$ is the local speed of sound. In subsonic flow the speed of sound is greater than a_∞ and the density is greater than ρ_∞; hence,

$$25\gamma p - \frac{(5-\theta^2)^2}{\rho} > 0 ,$$

so that the quantity in the outer brackets on the right-hand side of equation (2.22) is the sum of two positive terms. It follows that $D < 0$.

Equations (2.21) allow the introduction of a marching procedure in the direction of increasing s, starting from the line $s = 0$. Once the quantities b_1, e_1, f_1 and h_1 are known along a line of constant s, the derivatives of the quantities with respect to θ can be found numerically; equations (2.21) can then be used to determine the derivatives with respect to s; this allows another step to be taken in the direction of increasing s. The procedure can be started along the line $s = 0$ in the following way. When $s = 0$, equations (2.21b) and (2.21d) become

$$2(1-\theta^2)(f_o + \theta\, f_o')e_1 + 2[4(1-\theta\, e_o) + \theta(f_o + \theta\, f_o')]f_1 = -(8\, b_o + b_o') , \quad (2.23a)$$

$$2(1+\theta\, h_o)e_1 - 2(f_o + h_o)f_1 = \theta\, b_o ; \quad (2.23b)$$

from these equations $e_1(\theta, 0)$ and $f_1(\theta, 0)$ can be determined.

When $s = 0$, equations (2.21a) and (2.21c) become

$$4(1-\theta\ e_o)b_1 = [2(3-\theta^2)b_o - \theta(1-\theta^2)b_o']e_1 - \theta^2\ b_o'\ f_1$$
$$+\ 2\theta(1-\theta^2)b_o\ e_{1\theta} + 2\theta^2\ b_o\ f_{1\theta}\ , \qquad (2.23c)$$

$$(1-\theta\ e_o)h_1 = \gamma(1+\theta\ h_o)(f_1-\theta\ e_1)\quad ; \qquad (2.23d)$$

from these equations $b_1(\theta, 0)$ and $h_1(\theta, 0)$ can be determined. Differentiation of equations (2.21) with respect to s followed by substitution of zero for s allows the same procedure to be applied to obtain $b_{1s}(\theta, 0)$, $e_{1s}(\theta, 0)$ $f_{1s}(\theta, 0)$ and $h_{1s}(\theta, 0)$. Accordingly, the marching procedure can be started as soon as the function $b_o(\theta)$ has been prescribed.

This procedure differs from the one used in the problem of hypersonic flow past a blunt body[18]; there, the marching technique was effectively applied in the θ direction, from the line $\theta = 1$ (in Ref. 18 this represents the bow shock) towards the line $\theta = 0$. There is no obvious objection to solving the hypersonic problem by marching in the s direction instead; the reason for the choice of the θ direction is simply that this allows the shock shape to be prescribed, which is attractive from a physical point of view. On the other hand, it is impossible to solve the sonic problem by marching in the θ direction from the line $\theta = 0$. When $\theta = 0$, equations (2.21) become ordinary differential equations of the first order for b_1, e_1, f_1 and h_1; the only arbitrary quantity, $b_o(0)$, is simply an undetermined constant. Integration of the differential equations leads to functions b_1, e_1, f_1 and h_1 that are arbitrary only to the extent of containing five undetermined constants, $b_o(0)$ and four integration constants; in general, any one set of values for these five constants must correspond to an infinite number of bodies. In the hypersonic problem the dependent variables along the line are undetermined to within an arbitrary function, the shock shape, and there is no obvious reason why the correspondence between shock shape and body shape should not be one-to-one. The objection to solving the sonic problem by a marching procedure in the θ direction starting from the line $\theta = 0$ does not apply to one in the s direction starting from the line $s = 0$. From equations (2.23) it is seen that the quantities b_1, e_1, f_1 and h_1 are undetermined to the extent of an arbitrary function, $b_o(\theta)$; there is no obvious reason why the correspondence between $b_o(\theta)$ and the body shape should not be one-to-one.

$b_o(\theta)$ is not, in fact, entirely arbitrary; an examination of the results obtained in Section 2.1 shows that it must satisfy one minor restriction. The rth term $(r > 3)$ in the expansion on the right-hand side of equation (2.9) is a constant times τ^{r-3} times $g_{r-2}(\sigma)$, where, according to Euvrard[17], $g_r(\sigma)$ satisfies an ordinary linear differential equation of the second order; the physically relevant solution of this equation is a function that can always be obtained in closed form and that contains just one arbitrary constant; in other words, only one of the two complementary functions of the differential equation appears in the physically relevant solution. In general, this means that the functions of σ occurring in the expansions on the right-hand sides of equations (2.12), (2.13) and (2.15) can all be obtained in closed form and all contain one arbitrary constant. The only exceptions are the terms associated with $g_1(\sigma)$. From equation (2.11) it is seen that the relevant complementary function for $g_1(\sigma)$ is unity, so that the derivatives of this complementary function are zero. Hence, the terms in u, v, ψ and ζ associated with g_1 do not contain an arbitrary constant. Now, σ and τ can be obtained in terms of θ and s from equations (2.17), (2.13) and (2.8b); equations (2.12b) and (2.18b) can then be used to determine $b_o(\theta) + \theta s\, b_1(\theta, s)$; finally, $b_o(\theta)$ is found by putting s equal to zero. Because no arbitrary constant appears in the terms associated with g_1 in the expansions for u, v, ψ and ζ, it is found that $b_o'(0)$ depends only on $b_o(0)$; on the other hand, $b_o^{(r)}(0)$ can be arbitrarily prescribed when $r \geq 2$. The relation between $b_o'(0)$ and $b_o(0)$ is

$$b_o'(0) = -\frac{(9-2\gamma)}{3}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}} b_o \simeq -1.8866\, b_o \quad , \qquad (2.24)$$

on the assumption that $\gamma = 1.4$. Apart from this minor restriction $b_o(\theta)$ can be arbitrarily prescribed.

Another important point is that marching in the s direction cannot be continued indefinitely. This can be seen most clearly by considering the line $\theta = 1$, which corresponds to the body contour. From equation (2.17b), $s = y^2$ when $\theta = 1$; hence, $s < y_{max}^2$, where $2y_{max}$ is the maximum thickness of the body (referred to R_o). The part of the body lying beyond the position of maximum thickness is mapped into the portion of the line $\theta = 1$ lying between $s = y_{max}^2$ and $s = 0$. Consequently, along $\theta = 1$ the mapping of the x, y plane into the θ, s plane is not one-to-one; in particular, the point $\theta = 1$, $s = y_{max}^2$ must have a singularity associated with it. The same state of affairs exists along the line $\theta = 0$. From equations (2.12b), (2.18b), (2.17), (2.8b) and (2.13), it can be shown that along $\theta = 0$

$$s = \frac{2^{15/2}\,(\gamma+1)^{5/6}\,\mu^{10}\,\sigma(3-\sigma)^{5/2}\,(1+\sigma)^{5/2}}{3^{5/2}}\,, \qquad (2.25a)$$

$$b_o(0) + s\,b_1(0,\ s) = \frac{9(3-2\sigma)^2}{64(\gamma+1)^{4/3}\,\mu^{10}(3-\sigma)^4\,(1+\sigma)^4}\,. \qquad (2.25b)$$

Equation (2.25a) gives

$$\frac{ds}{d\sigma} = \frac{2^{15/2}\,(\gamma+1)^{5/6}\,\mu^{10}(3-2\sigma)(1+3\sigma)(3-\sigma)^{3/2}\,(1+\sigma)^{3/2}}{3^{5/2}}\,,$$

so that $\frac{ds}{d\sigma} = 0$ when $\sigma = 3/2$. It follows that $s = 0$ when $\sigma = 0$ and that s reaches a maximum when $\sigma = 3/2$; s_{max}, the maximum value of s, is given by

$$s_{max} = 2^{3/2}\,.\,3\,.\,5^{5/2}\,(\gamma+1)^{5/6}\,\mu^{10}\,. \qquad (2.26)$$

Hence, mapping of the x, y plane into the θ, s plane is not one-to-one along $\theta = 0$; in particular, the point $\theta = 0$, $s = s_{max}$ has a singularity associated with it. From equations (2.25) it can be shown that along $\theta = 0$ the expansion for $b_o(0) + s\,b_1(0,\ s)$ in the neighbourhood of the point $s = s_{max}$ starts with a term in $(s_{max} - s)$ followed by one in $(s_{max} - s)^{3/2}$. Later, it will be shown that a singular line of the mapping of x, y into θ, s runs from the point $\theta = 0$, $s = s_{max}$ to the point $\theta = 1$, $s = y^2_{max}$; only points to one side of this line (those having the smaller value of s) have physical significance, and such points correspond to two points in the physical plane.

The final step is the derivation of equations for returning from the θ, s plane to the x, y plane. From equations (2.5),

$$d\psi = -\rho v\,dx + \rho u\,dy\,. \qquad (2.27)$$

From equations (2.17a) and (2.27),

$$d\theta = \frac{\rho v}{2\theta y}\,dx + \frac{(1-\theta^2) - \rho u}{2\theta y}\,dy\,. \qquad (2.28)$$

From equation (2.17b),

$$ds = \frac{5s}{\theta}\,d\theta + \frac{2s}{y}\,dy\,. \qquad (2.29)$$

Along a line θ = constant equations (2.28) and (2.29) give

$$dx = -\frac{(1-\theta^2)-\rho u}{\rho v}\,dy = -\frac{y\left[\frac{(1-\theta^2)}{\rho} - u\right]}{2 s v}\,ds \quad .$$

From equations (2.17b), (2.18a), (2.18b), (2.18c) and (2.20a), it follows that

$$dx = -\frac{[\theta f_1 + (1-\theta^2)e_1]}{2\theta^2(b_o + s\, b_1)^{\frac{1}{2}}}\,ds \quad . \qquad (2.30)$$

Along a line s = constant equations (2.28) and (2.29) give

$$dx = \frac{y\left[(5-\theta^2)\frac{1}{\rho} - 5u\right]}{2\theta v}\,d\theta \quad .$$

From equations (2.17b), (2.18a), (2.18b), (2.18c) and (2.20a), it follows that

$$dx = \frac{[4(1-\theta\, e_o) + \theta s\, f_1 + (5-\theta^2)s\, e_1]}{2\theta^3\,(b_o + s\, b_1)^{\frac{1}{2}}}\,d\theta \quad . \qquad (2.31)$$

Equations (2.30) and (2.31) are equations for x; y can be obtained directly from equation (2.17b). For example, along the line $\theta = 1$

$$x = -\tfrac{1}{2}\int_o^s \frac{f_1(1,\, s)ds}{[b_o(1) + s\, b_1(1,\, s)]^{\frac{1}{2}}} \quad ,$$

$$y = s^{\frac{1}{2}} \quad ,$$

provided that the origin of axes is chosen to lie at the nose of the body; along the line s = 0

$$x = -2\int_\theta^1 \frac{(1-\theta\, e_o)d\theta}{\theta^3\, b_o^{\frac{1}{2}}} \quad ,$$

$$y = 0 \quad .$$

From equations (2.30) and (2.31) it can be seen that the mapping of x, y to s, θ becomes singular when $(b_o + s\, b_1) = 0$; from equation (2.18b), this occurs when v = 0. This agrees with the remarks made above about the singular points on the lines $\theta = 0$ and $\theta = 1$ and confirms that a singular line runs from the singular point on the line $\theta = 0$ to that on the line $\theta = 1$.

2.3 Axi-symmetric flow

It is not possible to extend the preceding technique to the solution of axi-symmetric sonic flow. This is best demonstrated by quickly going through the analysis and pointing out where the procedure breaks down.

The equations of motion and the boundary conditions are the same as for the two-dimensional problem, except that the continuity equation is now

$$\rho\, u_x + \rho\, v_y + \rho_x + v\, \rho_y - \frac{\rho v}{y} = 0 \quad . \tag{2.32}$$

The stream function must now be written as $\rho_\infty\, a_\infty\, R_o^2\, \psi$, where the non-dimensional quantity ψ satisfies the equations

$$\psi_y = \rho\, u y \quad , \tag{2.33a}$$

$$\psi_x = -\rho\, v y \quad ; \tag{2.33b}$$

elimination of ψ from equations (2.33) by cross-differentiation leads to equation (2.32). If ψ is chosen to be zero along the axis and on the body, then $\psi = y^2/2$ in the free stream. Hence, the variable ζ must now be defined by

$$\zeta = 1 - \frac{2\psi}{y^2} \quad , \tag{2.34}$$

instead of by equation (2.14).

It is known[16] that suitable independent variables, corresponding to those defined by equations (2.8), are σ and τ, where

$$x = -(\gamma+1)^{1/3}\, \mu\, \frac{(1-2\sigma)}{\tau^2} \quad , \tag{2.35a}$$

$$y = \sigma^{\frac{1}{2}}/\tau^{7/2} \quad , \tag{2.35b}$$

and that the expansions for ϕ, u and v in terms of τ start with

$$\phi = -(\gamma+1)^{1/3}\, \mu\, \frac{(1-2\sigma)}{\tau^2} - \frac{8\,\mu^3}{9}\, \tau(6+3\sigma-2\sigma^2) \quad , \tag{2.36}$$

$$u = 1 - \frac{8\,\mu^3}{3(\gamma+1)^{1/3}}\, (1-\sigma)\tau^3 \quad , \tag{2.37a}$$

$$v = \frac{16\,\mu^3}{9}\, \sigma^{\frac{1}{2}}\, (3-2\sigma)\tau^{9/2} \quad . \tag{2.37b}$$

From equations (2.37) and (2.7) the start of the expansion for ρ can be found. Equations (2.33) can then be used to determine the start of the expansion for ψ; this turns out to be

$$\psi = \frac{\sigma}{2\tau^7} - \frac{16(\gamma+1)^{1/3}\mu^4}{9}\frac{\sigma(1-\sigma)^2}{\tau} , \qquad (2.38)$$

where the first term is simply equal to $y^2/2$. From equations (2.34) and (2.38), the expansion for ζ starts with

$$\zeta = \frac{32(\gamma+1)^{1/3}\mu^4}{9}(1-\sigma)^2\tau^6 . \qquad (2.39)$$

From equation (2.35b), $\sigma = y^2\tau^7$; from equation (2.39) this means that σ varies as $\zeta^{7/6}$; this suggests writing $\zeta = \theta^6$, which corresponds to the substitution $\zeta = \theta^2$ in the two-dimensional problem. From equations (2.37) and (2.39), $(u-1)$ and v then vary as θ^3 and $\theta^{9/2}$ respectively; as in the two-dimensional problem, the final form of the equations of motion contains v only in the form v^2, which does not have a singularity when $\theta = 0$. Hence, the equations corresponding to equations (2.17) are

$$\theta = \left(1 - \frac{2\psi}{y^2}\right)^{1/6} , \qquad (2.40a)$$

$$s = y^2\theta^7 . \qquad (2.40b)$$

So far no difficulty has arisen in carrying out an analysis analogous to that of the two-dimensional problem. The next step, however, introduces what seems to be an insuperable obstacle. It is known[16] that the full expansion for ϕ is of the form

$$\phi = -(\gamma+1)^{1/3}\mu\frac{(1-2\sigma)}{\tau^2} + \mu^3\tau\left[-\frac{8}{9}(6+3\sigma-2\sigma^2) + \sum_{N=1}^{\infty}\tau^{\nu_N}g_N(\sigma) + \sum_{N=1}^{\infty}\tau^{\lambda_N}h_N(\sigma)\right] . \qquad (2.41)$$

Here, the g_N are functions of σ that can be obtained in simple closed form; each of them contains just one arbitrary constant. The h_N are functions of σ that can also be obtained in simple closed form; each of them contains some of the arbitrary constants present in the g_N but none of them contains new them are half the difference between a quadratic surd and an integer; examples of the indices are $\frac{\sqrt{145}-5}{2}$, 5, $\frac{\sqrt{481}-9}{2}$. From equations (2.40), (2.35) and (2.38) it follows that the expansion for ϕ (and, hence, for u, v, ρ and p) in powers of θ has a form similar to equation (2.41) in that a typical power of θ is occasionally an integer but is more usually half the difference between an integer and a quadratic surd. This is completely different from the two-dimensional problem, where the powers of θ are all integers. Hence, whereas the dependent variables in the two-dimensional problem can be expanded as Taylor series in powers of θ (the coefficients being functions of σ), the corresponding expansions in the axi-symmetric problem are most certainly not Taylor series. Further, there seems to be no obvious substitution to turn an expansion like that of equation (2.41) into a Taylor series. This means that any attempt at numerical differentiation in the θ direction is bound to fail, at least in the neighbourhood of the line $\theta = 0$; as this is an essential step in a marching procedure in the s direction, it follows that the axi-symmetric problem cannot be solved by such a procedure.

3 NUMERICAL RESULTS AND DISCUSSION

The marching procedure described in Section 2.2 has been programmed in Mercury Autocode. There are two programs, one for deriving starting values along the line s = 0, the other for marching from this line in the direction of increasing s.

The first program produces output consisting of the values of $e_o(\theta)$, $f_o(\theta)$, $h_o(\theta)$, $b_1(\theta, 0)$, $e_1(\theta, 0)$, $f_1(\theta, 0)$, $h_1(\theta, 0)$, $b_{1s}(\theta, 0)$, $e_{1s}(\theta, 0)$, $f_{1s}(\theta, 0)$ and $h_{1s}(\theta, 0)$ corresponding to $\theta = 0$, $\theta = 0.05$, ..., $\theta = 1$; it is assumed that $\gamma = 1.4$. The program has to be provided with a routine for calculating $b_o(\theta)$, $b_o'(\theta)$, $b_o''(\theta)$ and $b'''(\theta)$, since it can be shown that all these functions are required in the determination of the quantities $b_1(\theta, 0)$, ..., $h_{1s}(\theta, 0)$. The data tape for the first program contains the values at $\theta = 0$, $\theta = 0.05$, ..., $\theta = 1$ of $e_o(\theta)$, $f_o(\theta)$, $h_o(\theta)$, and various derivatives of these functions.

The second program has to be provided with a routine for calculating $b_o(\theta)$ and $b_o'(\theta)$. The output tape of the first program is the data tape for the second one, which also has to be given the length, δ, of the steps in the marching procedure.

The numerical techniques used in the marching procedure are about as simple as possible. Numerical differentiation of a function in the θ direction is performed by fitting a parabolic approximation to the function that passes through three successive points, those corresponding, say, to $\theta_A - 0.05$, θ_A, and $\theta_A + 0.05$; the differential coefficient of the approximation when $\theta = \theta_A$ is regarded as the differential coefficient of the function at that value of θ. The end points, $\theta = 0$ and $\theta = 1$, are treated in a similar way; for example, to obtain the differential coefficient when $\theta = 0$, θ_A is put equal to 0.05 and the differential coefficient of the approximation when $\theta = \theta_A - 0.05 = 0$ is taken. A step in the s direction from, say, the line $s = s_B$ to the line $s = s_B + \delta$ is taken as follows: the s derivatives of the dependent variables are found along the line $s = s_B$; a first approximation to the values of the dependent variables along the line $s = s_B + \delta$ is obtained by adding δ times the s derivatives along the line $s = s_B$ to the values of the dependent variables along the same line; the s derivatives along the line $s = s_B + \delta$ are found; a new approximation to the values of the dependent variables along the line $s = s_B$ is obtained by adding $\delta/2$ times the sum of the s derivatives along the lines $s = s_B$ and $s = s_B + \delta$ to the values of the dependent variables along the line $s = s_B$; the latter process is carried out twice more. The second program prints out the current value of s_B and the last two approximations of the dependent variables on the line $\theta = 1$, the line corresponding to the body contour; it would be easy to modify the program to print out the dependent variables for other values of θ if these were required.

The method has been applied to the flow past the body associated with the following choice of $b_o(\theta)$;

$$b_o(\theta) = 1 - 1.8866\,\theta + \theta^2 \quad . \qquad (3.1)$$

It is clear that this satisfies the restriction imposed by equation (2.24). Of course, it cannot do so exactly, since the number 1.8866 is only an approximation to $(9-2\gamma)\sqrt{2}/3\sqrt{(\gamma+1)}$; but there is no evidence that the slight error involved leads to divergence. The first singular point to be encountered is apparently the one lying on the line $\theta = 0$. Now, from equation (3.1), $b_o(0) = 1$, and, from equations (2.25),

$$b_o(0) = \frac{1}{64(\gamma+1)^{4/3}\,\mu^{10}} \quad .$$

Hence,

$$\mu^{10} = \frac{1}{64(\gamma + 1)^{4/3}} \quad ;$$

and, from equation (2.26),

$$s_{max} = \frac{3 \cdot 5^{5/2}}{2^{9/2} (\gamma + 1)^{\frac{1}{2}}} \simeq 4.784 \quad .$$

It follows that the marching procedure cannot be continued beyond the line $s = 4.784$ and that results obtained for values of s close to this line must be regarded with suspicion.

Figs. 1 and 2 show results obtained with a step-length, δ, of 0.02. Fig. 1 is a plot of p, obtained from equation (2.18d), against s from $s = 0$ to $s = 4.5$. The value of p at the sonic point is easily shown to be $1/\gamma$, so that the procedure has had to be stopped a long way short of the sonic point. Fig. 2 shows the body contour; here, x and y have been determined by the process described at the end of Section 2.2. Fig. 2 also contains a few values of the non-dimensional pressure, p. The curves in both Figs. 1 and 2 exhibit points of inflection that look a little odd, although they do not seem to have arisen from errors in the numerical procedure. Halving the step interval (0.01 instead of 0.02) makes no difference to the results. Further, as stated above, the last two approximations from the iteration procedure used in going from the line $s = s_B$ to $s = s_B + \delta$ are printed out; these always agree to five figures (at least, up to the point $s = 4.5$). In short, there seems little doubt that the marching procedure is stable.

It is all too obvious that the present investigation is still at a preliminary stage. Some means must be found of overcoming the problem of the singular line in the mapping from the x, y plane to the θ, s plane. In the example associated with equation (3.1) this might be done by cutting out the value $\theta = 0$ in the marching procedure shortly before the point $\theta = 0$, $s = s_{max}$ is reached and working only from $\theta = 0.05$ to $\theta = 1$. Presumably, a singular point on the line $\theta = 0.05$ would soon be reached; shortly before this happened, the value $\theta = 0.05$ would have to be cut out from the marching procedure as well as the value $\theta = 0$. By continuing this process it might be possible to reach the whole of the singular line rather than the point on it for which s is least. The next problem would be to determine the nature of the singularity at an arbitrary point on the singular line; this might well turn out to be simply a square root singularity, since it has been shown in Section 2.2 that this is so for the point $\theta = 0$, $s = s_{max}$. There might then be some means of marching backwards (in the direction of decreasing s), so as to obtain the values of the dependent variables in the second sheet of the θ, s plane. In any event, there would be no need to continue marching beyond the limiting characteristic[3], since conditions downstream of this could not affect points upstream of it; the remainder of the body could be arbitrarily prescribed and the flow field calculated by the method of characteristics, at least up to the shock wave that terminates the supersonic region.

One problem that might arise in the marching procedure is the vanishing of the determinant D defined by equation (2.22). This would presumably occur when a characteristic direction became parallel to either the θ or the s direction. There should not be any difficulty in dealing with this problem.

Choosing a function $b_o(\theta)$ that produces the required type of body is likely to be much more troublesome; from Fig.2 it is clear that the function defined by equation (3.1) has certainly not done so. The only way of solving this problem seems to be trial and error; experimentalists are unlikely to view with favour a request for values of the pressure on the line of symmetry from the free stream to the nose of the body, and these have to be known if b_o is to be determined from experimental results.

The author hopes to continue his investigation of the technique described in this paper, although his changed circumstances may make this difficult.

SYMBOLS

a	non-dimensional speed of sound
a_∞	free-stream speed of sound
b_o	arbitrary function of θ
b_1	see equation (2.18b)
D	determinant associated with equations (2.21a), (2.21b) and (2.21c)
e_o	defined by equation (2.20a)
e_1	see equation (2.18c)
f_o	defined by equation (2.19)
f_1	see equation (2.18a)
g_r	functions of σ in expansions of ϕ, ψ, etc in powers of τ
h_o	defined by equation (2.20b)
h_1	see equation (2.18d)
M_∞	free-stream Mach number
p	non-dimensional pressure
R_o	representative length
s	defined by equation (2.17b) and later by equation (2.40b)
s_{max}	maximum value of s on the line $\theta = 0$
U	free-stream speed
u, v	non-dimensional velocity components in the x and y directions respectively
X, Y, Z	rectangular cartesian coordinates
x, y	non-dimensional rectangular cartesian coordinates
y_{max}	half the maximum thickness of the body (referred to R_o)
γ	ratio of the specific heats of the fluid
δ	step length in marching procedure
ζ	defined by equation (2.14) and later by equation (2.34)
θ	defined by equation (2.17a) and later by equation (2.40a)
μ	scaling factor
ρ	non-dimensional density
ρ_∞	free-stream density
Σ	$\Psi/\rho_\infty UY$

SYMBOLS (Contd)

σ, τ	defined by equations (2.8) and later by equations (2.35)
Φ, ϕ	respectively actual and non-dimensional velocity potential
Ψ, ψ	respectively actual and non-dimensional stream function

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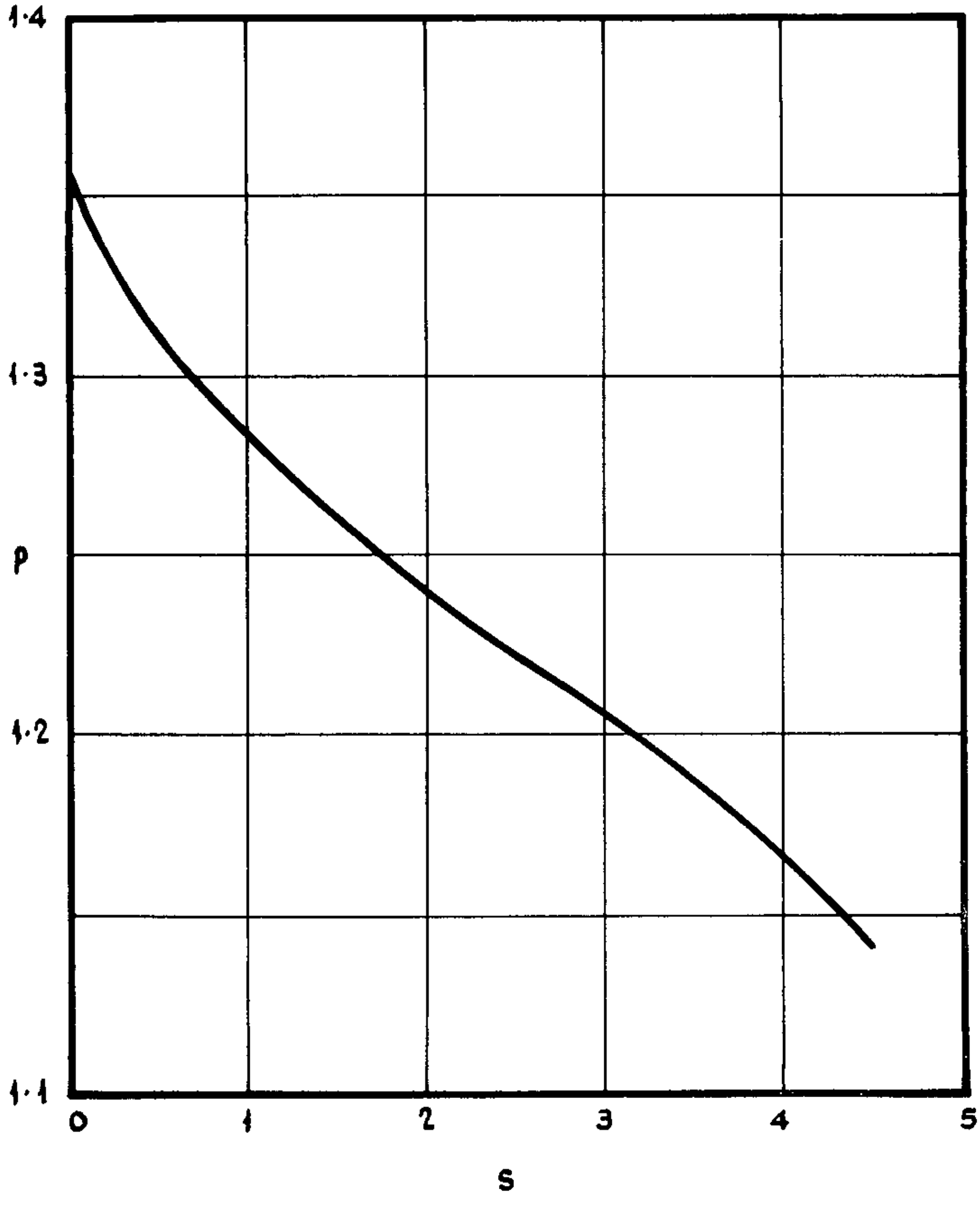

FIG. 1 NON-DIMENSIONAL PRESSURE ON THE LINE CORRESPONDING TO THE BODY CONTOUR

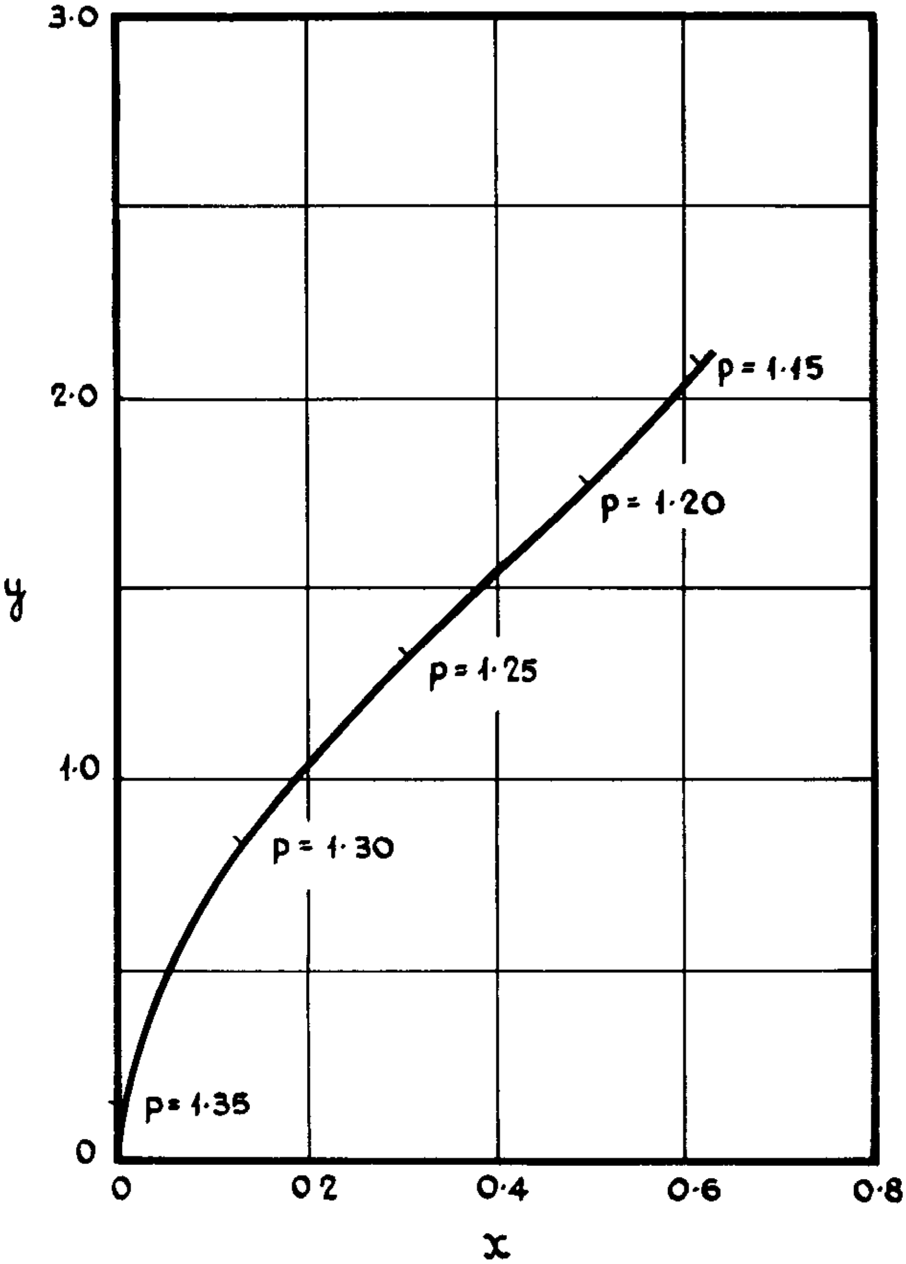

FIG. 2 THE BODY CONTOUR

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Randall, D.G.

A MARCHING PROCEDURE FOR THE DETERMINATION OF INVISCID TWO-DIMENSIONAL SONIC FLOW PAST A BLUNT SYMMETRICAL BODY

The equations of motion for two-dimensional, inviscid, sonic flow are written in a form that permits the introduction of a marching procedure for determining the flow past a symmetrical, blunt body. The independent variables are transformed to new variables, θ and s, such that points infinitely far from the body are mapped into the line $\theta = 0$, the line of symmetry is mapped into the line $s = 0$, and the body contour is mapped into the line $\theta = 1$; marching takes place in the s direction, starting from the line $s = 0$. The dependent variables are also transformed, in such a way that the physical quantities have the correct asymptotic behaviour far from the body.

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